changes to main text

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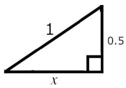
Placement exam

The answers[1] to this placement exam will tell you where to start reading.

1. What is the derivative of $\sin(x)$?

2. What is the second derivative of $A\sin(\omega x)$?

3. What is the value of x ?



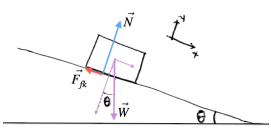
4. What is the magnitude of the gravitational force between two planets of mass M and mass m separated by a distance r?

5. Calculate $\displaystyle\lim_{x\to 3^-} \frac{1}{x-3}$.

6. Solve for t in:

$$7(3+4t) = 11(6t-4).$$

7. What is the component of the weight $\vec{W}$ acting in the x~~direction~~-direction?



8. A mass-spring system is undergoing simple harmonic motion. Its position function is $x(t) = A\sin(\omega t)$. What is its maximum acceleration?

[1] Ans: 1. $\cos(x)$, 2. $-A\omega^2\sin(\omega x)$, 3. $\frac{\sqrt{3}}{2}$, 4. $|\vec{F}_g| = \frac{GMm}{r^2}$, 5. $-\infty$, 6. $\frac{65}{38}$, 7. $+mg\sin\theta$, 8. $A\omega^2$. Key: If you didn't get Q3, Q6 right, you should read the book starting from Chapter 1. If you are mystified by Q1, Q2, Q5, read Chapter 5. If you want to learn how to solve Q4, Q7 and Q8, read Chapter 4.

Preface

This book contains lessons on topics in math and physics, written in a style that is jargon-free and to the point. Each lesson covers one concept at the depth required for a first-year university-level course. The main focus of this book is to highlight the intricate connections between the concepts of math and physics. Seeing the similarities and parallels between the concepts is the key to understanding.

Why?

The genesis of this book dates back to my student days when I was required to purchase expensive textbooks for my courses. Not only are these textbooks expensive, they are also tedious to read. Who has the energy to go through thousands of pages of explanations? I began to wonder, "What's the deal with these thick books?" Later, I realized mainstream textbooks are long because the textbook industry wants to make more profits. You don't need to read 1000 pages to learn calculus; the numerous full-page colour pictures and the repetitive text that are used to "pad" calculus textbooks are there to make the $130 price seem reasonable.

Looking at this situation, I said to myself, "something must be done," and I sat down and wrote a modern textbook to explain math and physics clearly, concisely, and affordably. There was no way I was going to let mainstream publishers ruin the learning experience of these beautiful subjects for the next generation.

How?

~~Each section~~ The sections in this book ~~is a~~ are **self-contained** ~~**tutorial**~~**tutorials**. Each section covers the definitions, formulas, and explanations associated with a single topic. You can therefore read the sections in any order you find logical. Along the way, you will learn about the *connections* between the concepts of calculus and mechanics. Understanding mechanics is much easier if you know the ideas of calculus. At the same time, the ideas behind calculus are best illustrated through concrete physics examples. Learning the two subjects simultaneously is the best approach.

~~To learn mechanics and calculus, you first need to know your high school math.~~ In order to make the ~~book~~ study of mechanics and calculus accessible for all readers, ~~the book begins~~ we'll begin with a review chapter on numbers, algebra, equations, functions, and other prerequisite concepts. If you feel a little rusty on those concepts, be sure to check out Chapter 1.

The end of each section contains links to interesting webpages, animations, and further reading material. You can use these links as a starting point for further exploration. The end of each chapter contains a series of exercises. Make sure you spend some quality time with them. You will learn a lot by solving exercises on your own.

Is this book for you?

My aim is to make learning calculus and mechanics more accessible. Anyone should be able to open this book and become proficient in calculus and mechanics, regardless of their mathematical background.

The book's primary intended audience is students. Students taking a mechanics class can read the chapters sequentially until Chapter 4, and optionally read Chapter 5 for general knowledge. Taking a calculus course? Skip ahead directly to the calculus chapter (Chapter 5). High school students or university students taking a precalculus class will benefit from reading Chapter 1, which is a concise but thorough review of fundamental math concepts like numbers, equations, functions, and trigonometry.

MECH CLASS	CALC CLASS	PRECALC CLASS
Ch. 1	Ch. 1	Ch. 1
Ch. 2	Ch. 2	Ch. 2†
Ch. 3		
Ch. 4		
Ch. 5†	Ch. 5	

† = optional reading.

Non-students, don't worry: you ~~do not~~ don't need to be taking a class in order to learn math. Independent learners interested in learning university-level material will find this book very useful. Many university graduates read this book to remember the calculus they learned back in their university days.

In general, anyone interested in rekindling ~~and improving~~ their relationship with mathematics should consider this book as an opportunity to repair the broken connection. Math is good stuff; you shouldn't miss out on it. People who think they absolutely *hate* math should read Chapter 1 as therapy.

In Chapter 2, we'll look at how techniques of high school math can be used to describe and model the world. We'll learn about the basic laws that govern the motion of objects in one dimension and the mathematical equations that describe the motion. By the end of this chapter, you'll be able to predict the flight time of a ball thrown in the air.

In Chapter 3, we will learn about vectors. Vectors describe directional quantities like forces and velocities. We need vectors to properly understand the laws of physics. Vectors are used in many areas of science and technology, so becoming comfortable with vector calculations will pay dividends when learning other subjects.

Chapter 4 is all about mechanics. We'll study the motion of objects, predict their future trajectories, and learn how to use abstract concepts like momentum and energy. Science students who "hate" physics can study this chapter to learn how to use the 20 main equations and laws of physics. You~~will~~'ll see physics is actually quite simple. Chapter 5 covers topics from differential calculus and integral calculus. We will study limits, derivatives, integrals, sequences, and series. You~~will~~'ll find that 100 pages are enough to cover all the concepts in calculus, as well as illustrate them with examples and practice exercises.

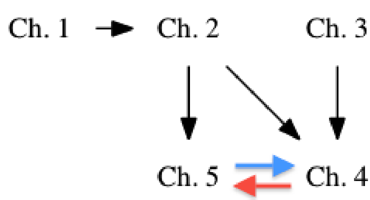


Figure 2: The prerequisite structure for the chapters in this book.

Calculus and mechanics are often taught as separate subjects. It shouldn't be like that! If you learn calculus without mechanics, it will be boring. If you learn physics without calculus, you won't truly understand. The exposition in this book covers both subjects in an integrated manner and aims to highlight the connections between them. Let's dig in.

which simplifies to

$$x^2 = 49.$$

The expression looks simpler, yes? How did I know to perform this operation? I was trying to "undo" the effects of the operation -4. We undo an operation by applying its *inverse*. In the case where the operation is subtraction of some amount, the inverse operation is the addition of the same amount. We'll learn more about function inverses in Section 1.4 (page 12).

We're getting closer to our goal, namely to *isolate* x on one side of the equation, leaving only numbers on the other side. The next step is to undo the square x^2 operation. The inverse operation of squaring a number x^2 is to take the square root $\sqrt{\ }$ so this is what we'll do next. We obtain

$$\sqrt{x^2} = \sqrt{49}.$$

Notice how we applied the square root to both sides of the equation? If we don't apply the same operation to both sides, we'll break the equality!

The equation $\sqrt{x^2} = \sqrt{49}$ simplifies to

$$|x| = 7.$$

What's up with the vertical bars around x? The notation $|x|$ stands for the *absolute value* of x, which is the same as x except we ignore the sign. For example $|5| = 5$ and $|-5| = 5$, too. The equation $|x| = 7$ indicates that both $x = 7$ and $x = -7$ satisfy the equation $x^2 = 49$. Seven squared is 49, and so is $(-7)^2 = 49$ because two negatives cancel each other out.

We're done since we isolated x. The final solutions are

$$x = 7 \qquad \text{or} \qquad x = -7.$$

Yes, there are *two* possible answers. You can check that both of the above values ~~satisfy our~~ of x satisfy the initial equation $x^2 - 4 = 45$.

If you are comfortable with all the notions of high school math and you feel you could have solved the equation $x^2 - 4 = 45$ on your own, then you should consider skipping ahead to Chapter 2. If on the other hand you are wondering how the squiggle killed the power two, then this chapter is for you! In the following sections we will review all the essential concepts from high school math that you will need to power through the rest of this book. First, let me tell you about the different kinds of numbers.

Variable names

There are common naming patterns for variables:

- x: general name for the unknown in equations (also used to denote a function's input, as well as an object's position in physics problems)
- v: velocity in physics problems
- θ, φ: the Greek letters *theta* and *phi* are ~~often~~ used to denote angles
- x_i, x_f: ~~denotes~~ denote an object's initial and final ~~position~~ positions in physics problems
- X: a random variable in probability theory
- C: costs in business along with P for profit, and R for revenue

Variable substitution

We can often *change variables* and replace one unknown variable with another to simplify an equation. For example, say you don't feel comfortable around square roots. Every time you see a square root, you freak out until one day you find yourself taking an exam trying to solve for x in the following equation:

$$\frac{6}{5-\sqrt{x}} = \sqrt{x}.$$

Don't freak out! In crucial moments like this, substitution can help with your root phobia. Just write, "Let $u = \sqrt{x}$" on your exam, and voila, you're allowed to rewrite the equation in terms of the variable u:

$$\frac{6}{5-u} = u,$$

which contains no square roots.

The next step to solve for u is to undo the division operation. Multiply both sides of the equation by $(5-u)$ to obtain

$$\frac{6}{5-u}(5-u) = u(5-u),$$

which simplifies to

$$6 = 5u - u^2.$$

This can be rewritten as a quadratic equation, $u^2 - 5u + 6 = 0$. Next, we can *factor* the quadratic to obtain the equation $(u-2)(u-3) = 0$, for which $u_1 = 2$ and $u_2 = 3$ are the solutions. The last step is to convert our u-answers into x answers by using $u = \sqrt{x}$, which is equivalent to $x = u^2$. The final answers are $x_1 = 2^2 = 4$ and $x_2 = 3^2 = 9$. Try plugging these x values into the original square root equation to verify that they satisfy it.

Compact notation

Symbolic manipulation is a powerful tool because it allows us to manage complexity. Say you're solving a physics problem in which you're told the mass of an object is $m = 140$ kg. If there are many steps in the calculation, would you rather use the number 140 kg in each step, or the shorter variable m? It's much easier in the long run to use the variable m throughout your calculation, and wait until the last step to substitute the value 140 kg when computing the final answer.

1.4 Functions and their inverses

As we saw in the section on solving equations, the ability to "undo" functions is a key skill for solving equations.

Example Suppose we're solving for x in the equation

$$f(x) = c,$$

where f is some function and c is some constant. Our goal is to isolate x on one side of the equation, but the function f stands in our way.

By using the inverse function (denoted f^{-1}) we "undo" the effects of f. Then we apply the inverse function f^{-1} to both sides of the equation to obtain

$$f^{-1}(f(x)) = x = f^{-1}(c).$$

By definition, the inverse function f^{-1} performs the opposite action of the function f so together the two functions cancel each other out. We have $f^{-1}(f(x)) = x$ for any number x.

Provided everything is kosher (the function f^{-1} must be defined for the input c), the manipulation we made above is valid and we have obtained the answer $x = f^{-1}(c)$.

The above example introduces the notation f^{-1} for denoting the function's *inverse*. This notation is borrowed from the notion of inverse numbers: multiplication by the number a^{-1} is the inverse operation of multiplication by the number a: $a^{-1}ax = 1x = x$. In the case of functions, however, the negative-one exponent does not refer to "one over-$f(x)$" as in $\frac{1}{f(x)} = (f(x))^{-1}$; rather, it refers to the function's inverse. In other words, the number $f^{-1}(y)$ is equal to the number x such that $f(x) = y$.

Be careful: sometimes applying the inverse leads to multiple solutions. For example, the function $f(x) = x^2$ maps two input values (x and $-x$) to the same output value $x^2 = f(x) = f(-x)$. The inverse function of $f(x) = x^2$ is $f^{-1}(x) = \sqrt{x}$, ~~and~~ but both $x = +\sqrt{c}$

Discussion

The recipe I have outlined above is not universally applicable. Sometimes x isn't alone on one side. Sometimes x appears in several places in the same equation. In these cases, you can't effortlessly work your way, Bruce Lee-style, clearing bad guys and digging toward x—you need other techniques.

The bad news is there's no general formula for solving complicated equations. The good news is the above technique of "digging toward x" is sufficient for 80% of what you are going to be doing. You can get another 15% if you learn how to solve the quadratic equation (page 19):

$$ax^2 + bx + c = 0.$$

Solving third-degree polynomial equations like $ax^3 + bx^2 + cx + d = 0$ with pen and paper is also possible, but at this point you might as well start using a computer to solve for the unknowns.

There are all kinds of other equations you can learn how to solve: equations with multiple variables, equations with logarithms, equations with exponentials, and equations with trigonometric functions. The principle of "digging" toward the unknown by applying ~~the function inverse~~ inverse functions is the key for solving all these types of equations, so be sure to practice using it.

1.5 Basic rules of algebra

It's important that you know the general rules for manipulating numbers and variables, a process otherwise known as—you guessed it—*algebra*. This little refresher will cover these concepts to make sure you're comfortable on the algebra front. We'll also review some important algebraic tricks, like *factoring* and *completing the square*, which are useful when solving equations.

When an expression contains multiple things added together, we call those things *terms*. Furthermore, terms are usually composed of many things multiplied together. When a number x is obtained as the product of other numbers like $x = abc$, we say "x factors into a, b, and c." We call a, b, and c the *factors* of x.

Given any four numbers a, b, c, and d, we can apply the following algebraic properties:

1. Associative property: $a + b + c = (a + b) + c = a + (b + c)$ and $abc = (ab)c = a(bc)$
2. Commutative property: $a + b = b + a$ and $ab = ba$
3. Distributive property: $a(b + c) = ab + ac$

Factoring the expression ~~x^2+5x+6~~ x^2-5x+6 will help us see the properties of the function more clearly. To *factor* a quadratic expression is to express it as the product of two factors:

$$f(x) = x^2 - 5x + 6 = (x-2)(x-3).$$

We now see at a glance the solutions (roots) are $x_1 = 2$ and $x_2 = 3$. We can also see for which x values the function will be overall positive: for $x > 3$, both factors will be positive, and for $x < 2$ both factors will be negative, and a negative times a negative gives a positive. For values of x such that $2 < x < 3$, the first factor will be positive, and the second factor negative, making the overall function negative.

For certain simple quadratics like the one above, you can simply *guess* what the factors will be. For more complicated quadratic expressions, you'll need to use the quadratic formula (page 19), which will be the subject of the next section. For now let us continue with more algebra tricks.

Completing the square

Any quadratic expression $Ax^2 + Bx + C$ can be rewritten in the form $A(x-h)^2 + k$ for some constants h and k. This process is called *completing the square* due to the reasoning we follow to find the value of k. The constants h and k can be interpreted geometrically as the horizontal and vertical shifts in the graph of the basic quadratic function. The graph of the function $f(x) = A(x-h)^2 + k$ is the same as the graph of the function $f(x) = Ax^2$ except it is shifted h units to the right and k units upward. We will discuss the geometrical meaning of h and k in more detail in Section 1.14 (page 58). For now, let's focus on the algebra steps.

Let's try to find the values of k and h needed to complete the square in the expression $x^2 + 5x + 6$. We start from the assumption that the two expressions are equal, and then expand the bracket to obtain

$$\underline{x^2}+5x+6 = A(x-h)^2+k = A(x^2-2hx+h^2)+k = \underline{Ax^2}-2Ahx+Ah^2+k.$$

Observe the structure in the above equation. On both sides of the equality there is one term which contains x^2 (the quadratic term), one term that contains x^1 (the linear term), and some constant terms. By focusing on the quadratic terms on both sides of the equation (they are underlined) we see $A = 1$, so we can rewrite the equation as

$$x^2 + \underline{5x} + 6 = x^2 \underline{-2hx} + h^2 + k.$$

Next we look at the linear terms (underlined) and infer $h = -2.5$. After rewriting, we obtain an equation ~~with a single~~ in which k is the only unknown:

$$x^2 + 5x + \underline{6} = x^2 - 2(-2.5)x + \underline{(-2.5)^2 + k}.$$

~~Finally, we~~ We must pick a value of k that ~~will make~~ makes the constant terms ~~match~~equal:

$$k = 6-(-2.5)^2 = 6-(2.5)^2 = 6-\left(\frac{5}{2}\right)^2 = 6\times\frac{4}{4}-\frac{25}{4} = \frac{24-25}{4} = \frac{-1}{4}.$$

After completing the square we obtain

$$x^2 + 5x + 6 = (x + 2.5)^2 - \frac{1}{4}.$$

The right-hand side of the expression above tells us our function is equivalent to the basic function x^2, shifted 2.5 units to the left and $\frac{1}{4}$ units down. This would be very useful information if you ever had to draw the graph of this function—you could simply plot the basic graph of x^2 and then shift it appropriately.

It is important you become comfortable with this procedure for completing the square. It is not extra difficult, but it does require you to think carefully about the unknowns h and k and to choose their values appropriately. There is no general formula for finding k, but you can remember the following simple shortcut for finding h. Given an equation $Ax^2 + Bx + C = A(x - h)^2 + k$, we have $h = \frac{-B}{2A}$. Using this shortcut will save you some time, but you will still have to go through the algebra steps to find k.

Take out a pen and a piece of paper now (yes, right now!) and verify that you can correctly complete the square in these expressions: $x^2 - 6x + 13 = (x - 3)^2 + 4$ and $x^2 + 4x + 1 = (x + 2)^2 - 3$.

1.6 Solving quadratic equations

What would you do if asked to solve for x in the quadratic equation $x^2 = 45x + 23$? This is called a *quadratic equation* since it contains the unknown variable x squared. The name comes from the Latin *quadratus*, which means square. Quadratic equations appear often, so mathematicians created a general formula for solving them. In this section, we'll learn about this formula and use it to put some quadratic equations in their place.

Before we can apply the formula, we need to rewrite the equation we are trying to solve in the following form:

$$ax^2 + bx + c = 0.$$

We reach this form—called the *standard form* of the quadratic equation—by moving all the numbers and xs to one side and leaving only 0 on the other side. For example, to transform the quadratic ~~expression~~ equation $x^2 = 45x + 23$ into standard form, subtract $45x + 23$ from both sides of the equation to obtain $x^2 - 45x - 23 = 0$. What are the values of x that satisfy this formula?

Claim

The solutions to the equation $ax^2 + bx + c = 0$ are

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad \text{and} \qquad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Let's see how these formulas are used to solve $x^2 - 45x - 23 = 0$. Finding the two solutions requires the simple mechanical task of identifying $a = 1$, $b = -45$, and $c = -23$ and plugging these values into the formulas:

$$x_1 = \frac{45 + \sqrt{45^2 - 4(1)(-23)}}{2} = 45.5054\ldots,$$

$$x_2 = \frac{45 - \sqrt{45^2 - 4(1)(-23)}}{2} = -0.5054\ldots.$$

Verify using your calculator that both of the values above satisfy the original equation $x^2 = 45x + 23$.

Proof of claim

This is an important proof. I want you to see how we can *derive* the quadratic formula from first principles because this knowledge will help you understand the formula. The proof will use the completing-the-square technique from the previous section.

Starting with $ax^2 + bx + c = 0$, first move c to the other side of the equation:

$$ax^2 + bx = -c.$$

Divide by a on both sides:

$$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

Now *complete the square* on the left-hand side by asking, "What are the values of h and k that satisfy ~~this~~ the equation

$$(x - h)^2 + k = x^2 + \frac{b}{a}x \quad ?"$$

Property 4 When an exponential expression is exponentiated, the inner exponent and the outer exponent multiply:

$$(b^m)^n = \underbrace{(\underbrace{bbb\cdots bb}_{m\text{ times}})(\underbrace{bbb\cdots bb}_{m\text{ times}})\cdots(\underbrace{bbb\cdots bb}_{m\text{ times}})}_{n\text{ times}} = b^{mn}.$$

Property 5.1

$$(ab)^n = \underbrace{(ab)(ab)(ab)\cdots(ab)(ab)}_{n\text{ times}} = \underbrace{aaa\cdots aa}_{n\text{ times}}\underbrace{bbb\cdots bb}_{n\text{ times}} = a^n b^n.$$

Property 5.2

$$\left(\frac{a}{b}\right)^n = \underbrace{\left(\frac{a}{b}\right)\left(\frac{a}{b}\right)\left(\frac{a}{b}\right)\cdots\left(\frac{a}{b}\right)\left(\frac{a}{b}\right)}_{n\text{ times}} = \frac{\overbrace{aaa\cdots aa}^{n\text{ times}}}{\underbrace{bbb\ \cdots\ bb}_{n\text{ times}}} = \frac{a^n}{b^n}.$$

Property 6 Raising a number to the power $\frac{1}{n}$ is equivalent to finding the n^{th} root of the number:

$$b^{\frac{1}{n}} \equiv \sqrt[n]{b}.$$

In particular, the square root corresponds to the exponent of one half: $\sqrt{b} = b^{\frac{1}{2}}$. The cube root (the inverse of x^3) corresponds to $\sqrt[3]{b} \equiv b^{\frac{1}{3}}$. We can verify the inverse relationship between $\sqrt[3]{x}$ and x^3 by using either Property 1: $\cancel{(\sqrt[3]{x})^3 = (x^{\frac{1}{3}})(x^{\frac{1}{3}})(x^{\frac{1}{3}}) = x^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = x^1 = x(\sqrt[3]{x})^3 = (x^{\frac{1}{3}})(x^{\frac{1}{3}})(}$ or by using Property 4: $(\sqrt[3]{x})^3 = (x^{\frac{1}{3}})^3 = x^{\frac{3}{3}} = x^1 = x$.

Properties 5.1 and 5.2 also apply for fractional exponents:

$$\sqrt[n]{ab} = (ab)^{\frac{1}{n}} = a^{\frac{1}{n}} b^{\frac{1}{n}} = \sqrt[n]{a}\,\sqrt[n]{b}, \qquad \sqrt[n]{\left(\frac{a}{b}\right)} = \left(\frac{a}{b}\right)^{\frac{1}{n}} = \frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}.$$

Discussion

Even and odd exponents

The function $f(x) = x^n$ behaves differently depending on whether the exponent n is even or odd. If n is odd we have

$$\left(\sqrt[n]{b}\right)^n = \sqrt[n]{b^n} = b.$$

However, if n is even, the function x^n destroys the sign of the number (see x^2, which maps both $-x$ and x to x^2). The successive application of light is written as `2.99792458e8` and the permeability of free space is `1.256637e-6`.

Links

[Further reading on exponentiation]
http://en.wikipedia.org/wiki/Exponentiation

[More details on scientific notation]
http://en.wikipedia.org/wiki/Scientific_notation

1.8 Logarithms

Some people think the word "logarithm" refers to some mythical, mathematical beast. Legend has it that logarithms are many-headed, breathe fire, and are extremely difficult to understand. Nonsense! Logarithms are simple. It will take you at most a couple of pages to get used to manipulating them, and that is a good thing because logarithms are used all over the place.

The strength of your sound system is measured in logarithmic units called decibels [dB]. This is because your ears are sensitive only to exponential differences in sound intensity. Logarithms allow us to compare very large numbers and very small numbers on the same scale. If sound were measured in linear units instead of logarithmic units, your sound system's volume control would need to range from 1 to 1 048 576. That would be weird, no? This is why we use the logarithmic scale for volume notches. Using a logarithmic scale, we can go from sound intensity level 1 to sound intensity level 1 048 576 in 20 "progressive" steps. Assume each notch doubles the sound intensity, rather than increasing the intensity by a fixed amount. If the first notch corresponds to 2, the second notch is 4—still probably inaudible, turn it up! By the time you get to the sixth notch you're at $2^6 = 64$ sound intensity, which is the level of audible music. The tenth notch corresponds to sound intensity $2^{10} = 1024$ (medium-strength sound), and finally the twentieth notch reaches a max power of $2^{20} = 1\,048\,576$, at which point the neighbours come knocking to complain.

Definitions

You~~are~~ 're hopefully familiar with these following concepts from the previous section:

- b^x: the exponential function base b
- $\exp(x) = e^x$: the exponential function base e, Euler's number

- 2^x: exponential function base 2
- $f(x)$: the notion of a function $f : \mathbb{R} \to \mathbb{R}$
- $f^{-1}(y)$: the inverse function of $f(x)$. It is defined in terms of $f(x)$ such that $f^{-1}(f(x)) = x$. In other words, if you apply f to some number and get the output y, and then you pass y through f^{-1}, the output will be x again. The inverse function f^{-1} undoes the effects of the function f.

In this section we~~will~~'ll play with the following new concepts:

- $\log_b(x)$: the logarithm of x base b is the inverse function of b^x.
- $\ln(x)$: the "natural" logarithm base e. This is the inverse of e^x.
- $\log_2(x)$: the logarithm base 2 is the inverse of 2^x.

I say *play* because there is nothing much new to learn here: a logarithm is a clever way to talk about the size of a number; essentially, it tells us how many digits the number has.

Formulas

The main thing to realize is that logs don't really exist on their own. They are defined as the inverses of their corresponding exponential functions. The following statements are equivalent:

$$\log_b(x) = m \qquad \Leftrightarrow \qquad b^m = x.$$

Logarithms with base e are written $\ln(x)$ for "logarithme naturel" because e is the "natural" base. Another special base is 10 because our numbers are based on the decimal system. The logarithm base 10 $\log_{10}(x)$ tells us roughly the size of the number x—how many digits the number has.

Example When someone working for the System (say someone with a high-paying job in the financial sector) boasts about his or her "six-figure" salary, they are really talking about the log of how much money they make. The "number of figures" N_S in their salary is calculated as 1 plus the logarithm base 10 of their salary S. The formula is

$$N_S = 1 + \log_{10}(S).$$

A salary of $S = 100\,000$ corresponds to $N_S = 1 + \log_{10}(100\,000) = 1+5 = 6$ figures. What is the smallest "seven-figure" salary? We must solve for S given $N_S = 7$ in the formula. We find $7 = 1 + \log_{10}(S)$, which means $6 = \log_{10}(S)$, and—using the inverse relationship between logarithm base 10 and exponentiation base 10—we discover $S = 10^6 = 1\,000\,000$. One million dollars per year! Yes, for this kind of money I see how someone might want to work for the System. But

~~Function names~~

Function names

We use short symbols like $+$, $-$, $\times$, and $\div$ to denote most of the important functions used in everyday life. We also use the weird *surd* notation to denote n^{th} root $\sqrt[n]{\ }$ and ~~the superscript notation~~ superscripts to denote exponents. All other functions are identified and denoted by their *name*. If I want to compute the *cosine* of the angle $60°$ (a function describing the ratio between the length of one side of a right-angle triangle and the hypotenuse), I write $\cos(60°)$, which means I want the value of the cos function for the input $60°$.

Incidentally, the function cos has a nice output value for that specific angle: $\cos(60°) \equiv \frac{1}{2}$. Therefore, seeing $\cos(60°)$ somewhere in an equation is the same as seeing $\frac{1}{2}$. To find other values of the function, say $\cos(33.13°)$, you'll need a calculator. A scientific calculator features a convenient little $\boxed{\cos}$ button for this very purpose.

Handles on functions

When you learn about functions you learn about the different "handles" by which you can "grab" these mathematical objects. The main handle for a function is its **definition**: it tells you the precise way to calculate the output when you know the input. The function definition is an important handle, but it is also important to "feel" what the function does intuitively. How does one get a feel for a function?

Table of values

One simple way to represent a function is to look at a list of input-output pairs: $\cancel{\{\{\text{in} = x_1, \text{out} = f(x_1)\},}$ $\{\{\text{in} = x_1, \text{out} = f(x_1)\},$ $\{\text{in} = x_2,$ $\cancel{\text{out} = f(x_2)\},}$ $\text{out} = f(x_2)\},$ $\{\text{in} = x_3, \text{out} = f(x_3)\}, \ldots\}$. A more compact notation for the input-output pairs is $\{(x_1, f(x_1)),$ $(x_2, f(x_2)),$ $(x_3, f(x_3)), \ldots\}$. You can make your own little **table of values**, pick some random inputs, and record the output of the function in the second column:

$$\begin{array}{rcl} \text{input} = x & \rightarrow & f(x) = \text{output} \\ 0 & \rightarrow & f(0) \\ 1 & \rightarrow & f(1) \\ 55 & \rightarrow & f(55) \\ x_4 & \rightarrow & f(x_4). \end{array}$$

In addition to choosing random numbers for your table, it's also generally a good idea to check the function's values at $x = 0$, $x = 1$,

Square root

The square root function is ~~defined as~~ denoted

$$f(x) = \sqrt{x} \equiv x^{\frac{1}{2}}.$$

The square root $\sqrt{x}$ is the inverse function of the ~~quadratic~~ square function x^2 for $x \geq 0$. The symbol $\sqrt{c}$ refers to the *positive* solution of $x^2 = c$. Note that $-\sqrt{c}$ is also a solution of $x^2 = c$.

Graph

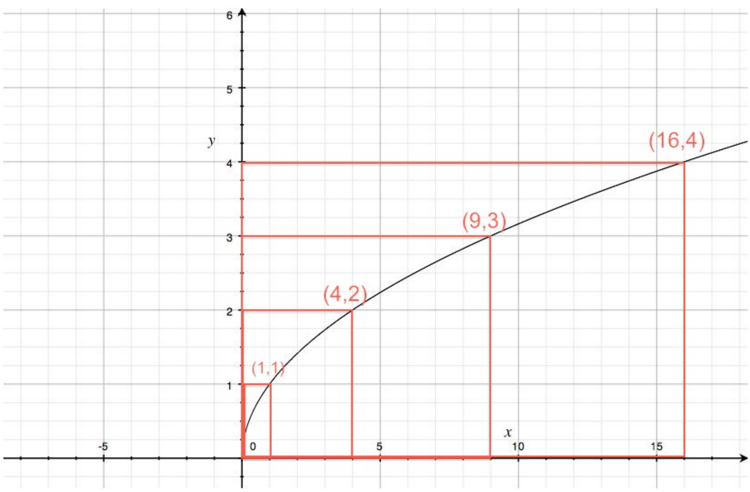


Figure 1.12: The graph of the function $f(x) = \sqrt{x}$. The domain of the function is $x \in [0, \infty$. You can't take the square root of a negative number.

Properties

- Domain: $x \in [0, \infty$.
  The function $f(x) = \sqrt{x}$ is only defined for nonnegative inputs $x \geq 0$. There is no real number y such that y^2 is negative, hence the function $f(x) = \sqrt{x}$ is not defined for negative inputs x.
- Image: $f(x) \in [0, \infty$.
  The outputs of the function $f(x) = \sqrt{x}$ are never negative: $\sqrt{x} \geq 0$, for all $x \in [0, \infty$.

In addition to *square* root, there is also *cube* root $f(x) = \sqrt[3]{x} \equiv x^{\frac{1}{3}}$, which is the inverse function for the cubic function $f(x) = x^3$. We have $\sqrt[3]{8} = 2$ since $2 \times 2 \times 2 = 8$. More generally, we can define the ~~root~~ n^{th} ~~root~~ -root function $\sqrt[n]{x}$ as the inverse function of x^n.

Polynomial functions

The general equation for a polynomial function of degree n is written,

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n.$$

The constants a_i are known as the *coefficients* of the polynomial.

Parameters

- n: the *degree* of the polynomial
- a_0: the constant term
- a_1: the *linear* coefficient, or first-order coefficient
- a_2: the *quadratic* coefficient
- a_3: the *cubic* coefficient
- a_n: the n^{th} order coefficient

A polynomial of degree n has $n+1$ coefficients: $a_0, a_1, a_2, \ldots, a_n$.

Properties

- Domain: $x \in \mathbb{R}$. Polynomials are defined for all inputs $x \in \mathbb{R}$.
- Image: depends on the coefficients
- The sum of two polynomials is also a polynomial.

Even and odd functions

The polynomials form an entire family of functions. Depending on the choice of degree n and coefficients a_0, a_1, …, a_n, a polynomial function can take on many different shapes. We~~will~~'ll study polynomials and their properties in more detail in Section 1.15, but for now consider the following observations about the symmetries of polynomials:

- If a polynomial contains only even powers of x, like $f(x) = 1 + x^2 - x^4$ for example, we call this polynomial *even*. Even polynomials have the property $f(x) = f(-x)$. The sign of the input doesn't matter.
- If a polynomial contains only odd powers of x, for example $g(x) = x + x^3 - x^9$, we call this polynomial *odd*. Odd polynomials have the property $g(x) = -g(-x)$.
- If a polynomial has both even and odd terms then it is neither even nor odd.

~~Note that the~~ The terminology of *odd* and *even* applies to functions in general and not just to polynomials. All functions ~~which~~ that satisfy $f(x) = f(-x)$ are called ~~even~~*even functions*, and all functions which satisfy $f(x) = -f(-x)$ are called ~~odd~~*odd functions*.

Exponential

The exponential function base $e = 2.7182818\ldots$ is denoted

$$f(x) = e^x \equiv \exp(x).$$

Graph

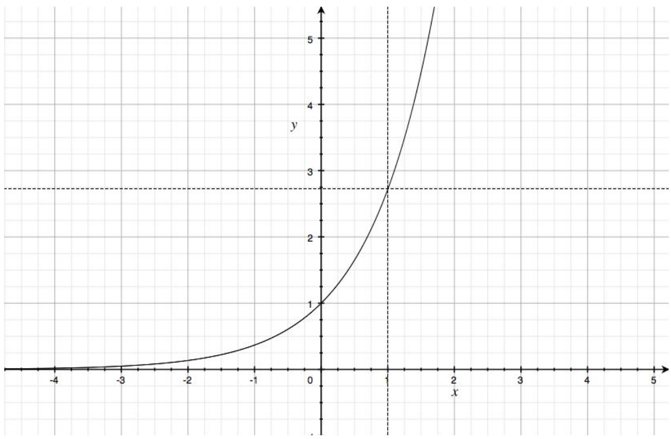


Figure 1.18: The graph of the exponential function $f(x) = e^x$ passes through the following (x, y) coordinates: $(-2, \frac{1}{e^2})$, $(-1, \frac{1}{e})$, $(0, 1)$, $(1, e)$, $(2, e^2)$, $(3, e^3 = 20.08\ldots)$, $(5, 148.41\ldots)$, and $(10, 22026.46\ldots)$.

Properties

- Domain: $x \in \mathbb{R}$
- Range: $e^x \in (0, \infty$
- $f(a)f(b) = f(a+b)$ since $e^a e^b = e^{a+b}$.
- The derivative (the slope of the graph) of the exponential function is equal to the exponential function: $f(x) = e^x \quad \Rightarrow \quad f'(x) = e^x$.

A more general exponential function would be $f(x) = Ae^{\gamma x}$, where A is the initial value, and γ (the Greek letter *gamma*) is the *rate* of the exponential. For $\gamma > 0$, the function $f(x)$ is increasing, as in Figure 1.18. For $\gamma < 0$, the function is decreasing and tends to zero for large values of x. The case $\gamma = 0$ is special since $e^0 = 1$, so $f(x)$ is a constant of $f(x) = A1^x = A$.

Links

[The exponential function 2^x evaluated]
http://www.youtube.com/watch?v=e4MSN6IImpI

namely $x = -2$ and $x = 5$, also satisfy

$$2x^2 + 2x = x^2 + 5x + 10,$$

which is the original problem we're trying to solve.

This "shuffling of terms" approach will work for any polynomial equation $A(x) = B(x)$. We can always rewrite it as $C(x) = 0$, where $C(x)$ is a new polynomial with coefficients equal to the difference of the coefficients of A and B. Don't worry about which side you move all the coefficients to because $C(x) = 0$ and $0 = -C(x)$ have exactly the same solutions. Furthermore, the degree of the polynomial C can be no greater than that of A or B.

The form $C(x) = 0$ is the *standard form* of a polynomial, and we'll explore several formulas you can use to find its solution(s).

Formulas

The formula for solving the polynomial equation $P(x) = 0$ depends on the *degree* of the polynomial in question.

First

For a first-degree polynomial equation, $P_1(x) = mx + b = 0$, the solution is $\cancel{x = \frac{b}{m}}$ $x = \frac{-b}{m}$: just move b to the other side and divide by m.

Second

For a second-degree polynomial,

$$P_2(x) = ax^2 + bx + c = 0,$$

the solutions are $x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$.

If $b^2 - 4ac < 0$, the solutions will involve taking the square root of a negative number. In those cases, we say no real solutions exist.

Higher degrees

There is also a formula for polynomials of degree 3, but it is complicated. For polynomials with order ≥ 5, there does not exist a general analytical solution.

Using a computer

When solving real-world problems, you'll often run into much more complicated equations. To find the solutions of anything more complicated than the quadratic equation, I recommend using a computer algebra system like `sympy`: `http://live.sympy.org`.

Pythagoras' theorem

In a right-angle triangle, the length of the hypotenuse squared is equal to the sum of the squares of the lengths of the other sides:

$$|\text{adj}|^2 + |\text{opp}|^2 = |\text{hyp}|^2.$$

If we divide both sides of the above equation by $|\text{hyp}|^2$, we obtain

$$\frac{|\text{adj}|^2}{|\text{hyp}|^2} + \frac{|\text{opp}|^2}{|\text{hyp}|^2} = 1,$$

which can be rewritten as

$$\cos^2\theta + \sin^2\theta = 1.$$

This is a powerful *trigonometric identity* ~~describing the~~ that describes an important relationship between sin and cos.

Sin and cos

Meet the trigonometric functions, or trigs for short. These are your new friends. Don't be shy now, say hello to them.

"Hello."

"Hi."

"Soooooo, you are like functions right?"

"Yep," sin and cos reply in chorus.

"Okay, so what do you do?"

"Who me?" asks cos. "Well I tell the ratio... hmm... Wait, are you asking what I do as a *function* or specifically what *I* do?"

"Both I guess?"

"Well, as a function, I take angles as inputs and I give ratios as answers. More specifically, I tell you how 'wide' a triangle with that angle will be," says cos all in one breath.

"What do you mean wide?" you ask.

"Oh yeah, I forgot to say, the triangle must have a hypotenuse of length 1. What happens is there is a point P that moves around on a circle of radius 1, and we *imagine* a triangle formed by the point P, the origin, and the point on the x-axis located directly below the point P."

"I am not sure I get it," you confess.

"Let me try explaining," says sin. "Look on the next page, and you'll see a circle. This is the unit circle because it has a radius of 1. You see it, yes?"

"Yes."

"This circle is really cool. Imagine a point P that starts from the point $P(0) = (1, 0)$ and moves along the circle of radius 1. The x and

y ~~coordinates~~ coordinates of the point $P(\theta) = (P_x(\theta),\ P_y(\theta))$ as a function of θ are

$$P(\theta) = (P_x(\theta),\ P_y(\theta)) = (\cos\theta,\ \sin\theta).$$

So, *either* you can think of us in the context of triangles, or you think of us in the context of the unit circle."

"Cool. I kind of get it. Thanks so much," you say, but in reality you are weirded out. Talking functions? "Well guys. It was nice to meet you, but I have to get going, to finish the rest of the book."

"See you later," says cos.

"Peace out," says sin.

The unit circle

The unit circle consists of all points (x, y) that satisfy the equation $x^2 + y^2 = 1$. A point $P = (P_x, P_y)$ on the unit circle has coordinates $(P_x, P_y) = (\cos\theta, \sin\theta)$, where θ is the angle P makes with the x-axis.

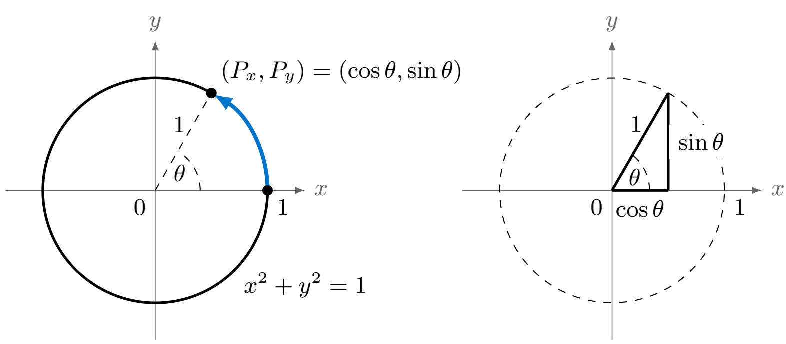


Figure 1.22: The unit circle corresponds to the equation $x^2 + y^2 = 1$. The coordinates of the point P on the unit circle ~~$(\cos\theta, \sin\theta)$~~ are ~~indicated for several important values of the angle θ~~ $P_x = \cos\theta$ and $P_y = \sin\theta$.

You should be familiar with the values of sin and cos for all angles that are multiples of $\frac{\pi}{6}$ (30°) or $\frac{\pi}{4}$ (45°). All of them are shown in Figure 1.24. For each angle, the x-coordinate (the first number in the bracket) is ~~cos~~$\cos\theta$, and the y-coordinate is ~~sin~~$\sin\theta$.

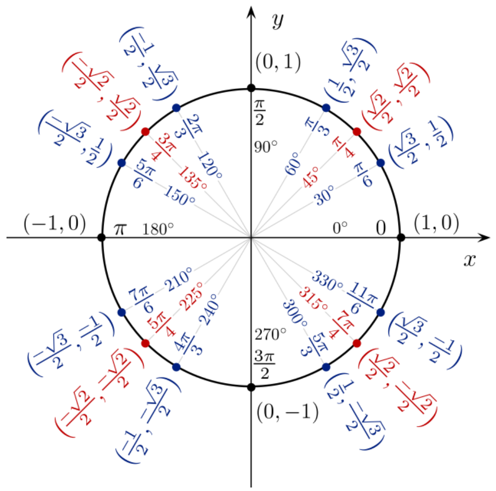


Figure 1.24: The unit circle. The coordinates of the point on the unit circle $(\cos\theta, \sin\theta)$ are indicated for several important values of the angle θ.

Maybe you're thinking that's way too much to remember. Don't worry, you just have to memorize one fact:

$$\sin(30°) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}.$$

Knowing this, you can determine all the other angles. Let's start with $\cos(30°)$. We know that at $30°$, point P on the unit circle has the vertical coordinate $\frac{1}{2} = \sin(30°)$. We also know the cos quantity we are looking for is, by definition, the horizontal component:

$$P = (\cos(30°), \sin(30°)).$$

Key fact: all points on the unit circle are a distance of 1 from the origin. Knowing that P is a point on the unit circle, and knowing the

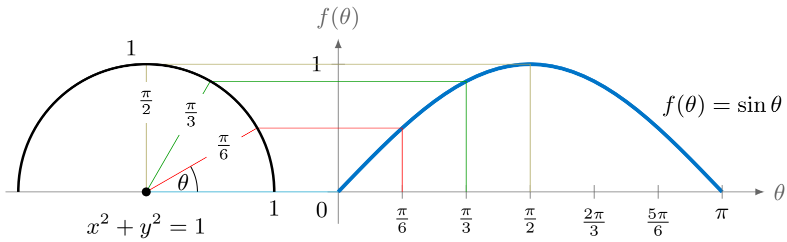


Figure 1.23: The function $f(\theta) = \sin\theta$ describes the vertical position of a point P that travels along the unit circle. The first half of a cycle is shown.

From this point on in the book, we'll always talk about the length of the *adjacent* side as $r_x = r\cos\theta$, and the length of the *opposite* side as $r_y = r\sin\theta$. It is extremely important you get comfortable with this notation.

The reasoning behind the above calculations is as follows:

$$\cos\theta \equiv \frac{\text{adj}}{\text{hyp}} = \frac{r_x}{r} \quad \Rightarrow \quad r_x = r\cos\theta,$$

and

$$\sin\theta \equiv \frac{\text{opp}}{\text{hyp}} = \frac{r_y}{r} \quad \Rightarrow \quad r_y = r\sin\theta.$$

Calculators

Make sure to set your calculator to the correct units for working with angles. What should you type into your calculator to compute the sine of 30 degrees? If your calculator is set to degrees, simply type: `30`, `sin`, `=`.

If your calculator is set to radians, you have two options:

1. Change the `mode` of the calculator so it works in degrees.
2. Convert 30° to radians

$$30\ [^\circ] \times \frac{2\pi\ [\text{rad}]}{360\ [^\circ]} = \frac{\pi}{6}\ [\text{rad}],$$

and type: `π`, `/`, `6`, `sin`, `=` on your calculator.

Links

[Unit-circle walkthrough and tricks by patrickJMT on YouTube]
bit.ly/1mQg9Cj and bit.ly/1hvA702

1.17 Trigonometric identities

There are a number of important relationships between the values of the functions sin and cos. Here are three of these relationships, known as *trigonometric identities*. There about a dozen other identities that are less important, but you should memorize these three.

The three identities to remember are:

1. Unit hypotenuse

$$\sin^2(\theta) + \cos^2(\theta) = 1.$$

The unit hypotenuse identity is true by the Pythagoras theorem and the definitions of sin and cos. The ~~ratio~~ sum of the squares of the sides of a triangle ~~are~~ is equal to the square of the ~~size of the~~ hypotenuse.

2. sico + sico

$$\sin(a + b) = \sin(a)\cos(b) + \sin(b)\cos(a).$$

The mnemonic for this identity is "sico + sico."

3. coco − sisi

$$\cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b).$$

The mnemonic for this identity is "coco - sisi." The negative sign is there because it's not good to be a sissy.

Derived formulas

If you remember the above three formulas, you can derive pretty much all the other trigonometric identities.

Double angle formulas

Starting from the sico-sico identity as explained above, and setting $a = b = x$, we can derive the following identity:

$$\sin(2x) = 2\sin(x)\cos(x).$$

Starting from the coco-sisi identity, we derive ~~or, if we rewrite this equation~~

$$\begin{aligned} \cos(2x) &= \cos^2(x) - \sin^2(x) \\ &= 2\cos^2(x) - 1 = 2\left(1 - \sin^2(x)\right) - 1 = 1 - 2\sin^2(x). \end{aligned}$$

The formulas for expressing $\sin(2x)$ and $\cos(2x)$ in terms of $\sin(x)$ and $\cos(x)$ are called *double angle formulas*.

If we rewrite the double-angle formula for $\cos(2x)$ to isolate the $\sin^2$ ~~and~~ or the $\cos^2$ ~~, we get~~ term, we obtain the *power-reduction formulas*:

$$\cos^2(x) = \frac{1}{2}\left(1 + \cos(2x)\right), \qquad \sin^2(x) = \frac{1}{2}\left(1 - \cos(2x)\right).$$

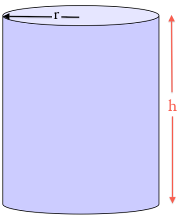
Cylinder

The surface area of a cylinder consists of the top and bottom circular surfaces, plus the area of the side of the cylinder:

$$A = 2\left(\pi r^2\right) + (2\pi r)h.$$

The formula for the volume of a cylinder is the product of the area of the cylinder's base times its height:

$$V = \left(\pi r^2\right) h.$$



Example You open the hood of your car and see 2.0 L written on top of the engine. The 2.0 L refers to the total volume of the four pistons, which are cylindrical in shape. The owner's manual tells you the diameter of each piston (bore) is 87.5 mm, and the height of each piston (stroke) is 83.1 mm. Verify that the total volume of the cylinder displacement of your engine is indeed 1998789 $\text{mm}^3 \approx 2$ L.

~~Links~~

~~Formula for calculating the distance between two points on a sphere~~

1.19 Circle

The circle is a set of points located a constant distance from a centre point. This geometrical shape appears in many situations.

Definitions

- r: the radius of the circle
- A: the area of the circle
- C: the circumference of the circle
- (x, y): a point on the circle
- θ: the angle (measured from the x-axis) of some point on the circle

Formulas

A circle with radius r centred at the origin is described by the equation

$$x^2 + y^2 = r^2.$$

All points (x, y) that satisfy this equation are part of the circle.

Try to visualize the curve traced by the point $(x(\theta), y(\theta)) = (r\cos\theta, r\sin\theta)$ as θ varies from $0°$ to $360°$. The point will trace out a circle of radius r.

If we let the parameter θ vary over a smaller interval, we'll obtain subsets of the circle. For example, the parametric equation for the top half of the circle is

$$\{(x, y) \in \mathbb{R}^2 \mid x = r\cos\theta, y = r\sin\theta,\ \theta \in [0, 180°]\}.$$

The top half of the circle is also described by $\{(x, y) \in \mathbb{R}^2 \mid y = \sqrt{r^2 - x^2},\ x \in [-r, r]\}$, where the parameter used is the x-coordinate.

Area

The area of a circle of radius r is $A = \pi r^2$.

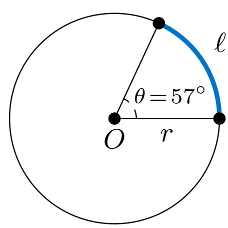
Circumference and arc length

The circumference of a circle is

$$C = 2\pi r.$$

This is the total length you can measure by following the curve all the way around to trace the outline of the entire circle.

What is the length of a part of the circle? Say you have a piece of the circle, called an *arc*, and that piece corresponds to the angle $\theta = 57°$. What is the arc's length ~~?~~ ℓ?



If the circle's total length $C = 2\pi r$ represents a full $360°$ turn around the circle, then the arc length ℓ for a portion of the circle corresponding to the angle θ is

$$\ell = 2\pi r \frac{\theta}{360}.$$

~~Note the~~ The arc length ℓ depends on r, the angle θ, and a factor of $\frac{2\pi}{360}$.

Radians

Though degrees are commonly used as a measurement unit for angles, it~~is~~'s much better to measure angles in *radians*, since radians are the *natural* units for measuring angles. The conversion ratio ~~from degrees to radians is For a circle of radius $r = 1$~~between degrees and radians is

$$2\pi[\text{rad}] = 360°.$$

When measuring angles in radians, the arc length is ~~equal to the angle in radians: Measuring~~ given by:

$$\ell = r\theta_{\text{rad}}.$$

Measuring angles in radians is equivalent to measuring arc length on a circle ~~of radius 1.~~ with radius $r = 1$.

1.20 Ellipse

The ellipse is a fundamental shape that occurs in nature. The orbit of planet Earth around the Sun is an ellipse.

Parameters

- a: the half-length of the ellipse along the x-axis, also known as the semi-major axis
- b: the half-length of the ellipse along the y-axis
- ε: the *eccentricity* of the ellipse, $\cancel{\varepsilon \equiv \sqrt{1 - \frac{b^2}{a^2}}}$ $\varepsilon \equiv \sqrt{1 - \frac{b^2}{a^2}}$
- F_1, F_2: the two *focal points* of the ellipse
- r_1: the distance from a point on the ellipse to F_1
- r_2: the distance from a point on the ellipse to F_2

Definition

An ellipse is the curve found by tracing along all the points for which the sum of the distances to the two focal points is a constant:

$$r_1 + r_2 = \text{const}.$$

There's a neat way to draw a perfect ellipse using a piece of string and two tacks or pins. Take a piece of string and tack it to a picnic table at two points, leaving some loose slack in the middle of the string. Now take a pencil, and without touching the table, use the pencil to pull the middle of the string until it is taut. Make a mark at that point. With the two parts of string completely straight, make a mark at every point possible where the two "legs" of string remain taut.

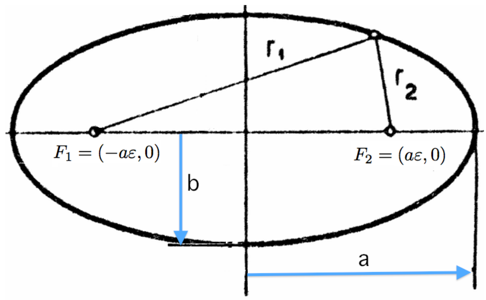


Figure 1.25: An ellipse with semi-major axis a and semi-minor axis b. The locations of the focal points F_1 and F_2 are indicated.

An ellipse is a set of points (x, y) that satisfy the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The *eccentricity* of an ellipse describes how elongated it is:

$$\varepsilon \equiv \sqrt{1 - \frac{b^2}{a^2}}\,.$$

The parameter $\varepsilon \in [0, 1)$ describes the *shape* of the ellipse in a scale-less fashion. The bigger ε is, the bigger the difference will be between the length of the semi-major axis and the semi-minor axis. In the special case when $\varepsilon = 0$, the equation of the ellipse becomes a circle with radius a.

The (x, y)-coordinates of the two focal points are

$$F_1 = (-a\varepsilon, 0) \qquad \text{and} \qquad F_2 = (a\varepsilon, 0).$$

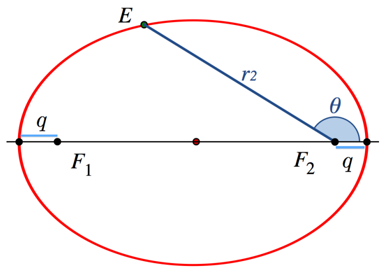
The focal points correspond to the locations of the two tacks where the string is held in place. Recall that we defined the variables r_1 and r_2 to represent the distance from the focal points F_1 and F_2. Furthermore, we will denote by $q = a(1 - \varepsilon)$ the distance of the ellipse's closest approach to a focal point.

~~Polar coordinates~~

Polar coordinates

In polar coordinates, the ellipse can be described by a function $r_2(\theta)$. This function gives the distance of a point E from F_2 as a function of the angle θ. Recall in polar coordinates, the angle θ is the independent variable and the dependent variable is the distance $r_2(\theta)$.

The equation of the ellipse in polar coordinates depends on the length of the semi-major axis a and the eccentricity ε. The equation that describes an ellipse in polar coordinates is



$$r_2(\theta) = \frac{a(1-\varepsilon^2)}{1+\varepsilon\cos(\theta)},$$

where the angle θ is measured with respect to the positive x-axis. The distance is smallest when $\theta = 0$ with $r_2(0) = a(1-\varepsilon) = q$ and largest when $\theta = \pi$ with $r_2(\pi) = a + a\varepsilon = a(1+\varepsilon)$.

Calculating the orbit of the Earth

~~Calculating the orbit of the Earth~~

To a close approximation, the motion of the Earth around the Sun is described by an ellipse with the Sun positioned at the focus F_2. We can therefore use the polar coordinates formula $r_2(\theta)$ to describe the distance of the Earth from the Sun.

The eccentricity of Earth's orbit around the Sun is $\varepsilon = 0.01671123$, and the half-length of the major axis is $a = 149\,598\,261$[km]. We substitute these values into the general formula for $r_2(\theta)$ and obtain the following equation:

$$r_2(\theta) = \frac{149\,556\,484.56}{1+0.01671123\cos(\theta)} \quad [\text{km}].$$

The point where the Earth is closest to the Sun is called the *perihelion*. It occurs when $\theta = 0$, which happens around the 3rd of January. The moment where the Earth is most distant from the Sun is called the *aphelion* and corresponds to the angle $\theta = \pi$. Earth's *aphelion* happens around the 3rd of July.

We can use the formula for $r_2(\theta)$ to predict the *perihelion* and *aphelion* distances of Earth's orbit:

$$r_{2,\text{peri}} = r_2(0) = \frac{149556483}{1+0.01671123\cos(0)} = 147\,098\,290 \ [\text{km}],$$

$$r_{2,\text{aphe}} = r_2(\pi) = \frac{149556483}{1+0.01671123\cos(\pi)} = 152\,098\,232 \ [\text{km}].$$

~~Can you~~ Google "perihelion" and "aphelion" to verify that the above predictions are accurate~~?~~ .

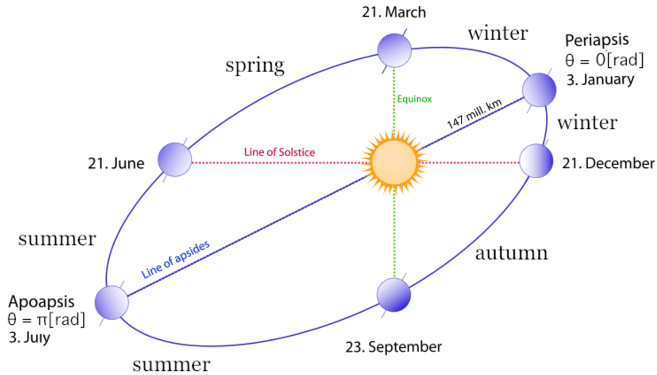


Figure 1.26: The orbit of the Earth around the Sun. Key points of the orbit are labelled. The seasons in the Northern hemisphere are also indicated.

The angle θ of the Earth relative to the Sun can be described as a function of time $\theta(t)$. The exact formula of the function $\theta(t)$ that describes the angle as a function of time is fairly complicated, so we won't go into the details. Let's simply look at some values of $\theta(t)$ with t measured in days. We'll begin on Jan 3^{rd}.

t [day]	1	2	.	182	.	365	365.242199
t [date]	Jan 3	Jan 4	.	July 3	.	Jan 2	?
$\theta(t)$ [°]	0		.	180	.	359.761356	360
$\theta(t)$ [rad]	0		.	π	.	6.27902	2π

Table 1.1: The angular position of the Earth as a function of time. Note the extra amount of "day" that is roughly equal to $\frac{1}{4} = 0.25$. We account for this discrepancy by adding an extra day to the calendar once every four years.

~~Newton's insight~~

Newton's insight

Contrary to common belief, Newton did not discover his theory of gravitation because an apple fell on his head while sitting under a tree. What actually happened is that he started from Kepler's laws of motion, which describe the exact elliptical orbit of the Earth as a function of time. Newton asked, "What kind of force would cause two bodies to spin around each other in an elliptical orbit?" He determined that the gravitational force between the Sun of mass M and the Earth of mass m must be of the form $F_g = \frac{GMm}{r^2}$. We'll discuss more about the law of gravitation in Chapter 4.

For now, let's give props to Newton for connecting the dots, and props to Johannes Kepler for studying the orbital periods, and Tycho Brahe for doing all the astronomical measurements. Above all, we owe some props to the ellipse for being such an awesome shape!

By the way, the varying distance between the Earth and the Sun is not the reason we have seasons. The ellipse had nothing to do with seasons! Seasons are predominantly caused by the *axial tilt* of the Earth. The axis of rotation of the Earth is tilted by 23.4° relative to the plane of its orbit around the Sun. In the Northern hemisphere, the longest day of the year is the summer solstice, which occurs around the 21$^{\text{th}}$ of June. On that day, the Earth's spin axis is tilted toward the Sun so the Northern hemisphere receives the most sunlight.

Links

[Further reading about Earth-Sun geometry]
http://www.physicalgeography.net/fundamentals/6h.html

1.21 Hyperbola

The hyperbola is another fundamental shape of nature. A horizontal hyperbola is the set of points (x, y) which satisfy the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

The numbers a and b are arbitrary constants. This hyperbola passes through the points $(-a, 0)$ and $(a, 0)$. The eccentricity of this hyperbola is defined as

$$\varepsilon = \sqrt{1 + \frac{b^2}{a^2}}\,.$$

Eccentricity is an important parameter of the hyperbola, as it determines the hyperbola's shape. Recall the ellipse is also defined by an eccentricity parameter, though the formula is slightly different. This could be a coincidence—or is there a connection? Let's see.

Graph

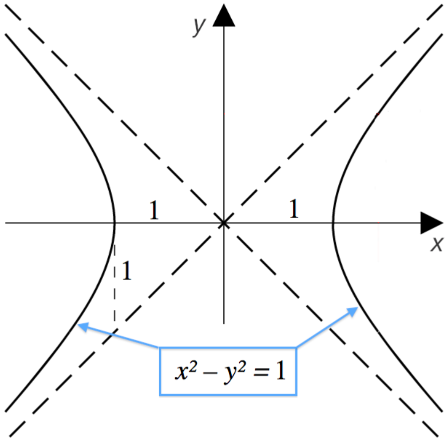


Figure 1.27: The unit hyperbola $x^2 - y^2 = 1$. The graph of the hyperbola has two branches, opening to the sides. The dashed lines are called the *asymptotes* of the hyperbola. The eccentricity determines the angle between the asymptotes. The eccentricity of $x^2 - y^2 = 1$ is $\varepsilon = \sqrt{1 + \frac{1}{1}} = \sqrt{2}$.

The graph of a hyperbola consists of two ~~separated~~ separate *branches*, as illustrated in Figure 1.27. We'll focus our discussion mostly on the right branch of the hyperbola.

~~**Hyperbolic trigonometry**~~

Hyperbolic trigonometry

The trigonometric functions sin and cos describe the geometry of the unit circle. The point $P = (\cos\theta, \sin\theta)$ traces out the unit circle as the angle θ goes from 0 to 2π. The function cos is defined as the x-coordinate of the point P, and sin is the y-coordinate. The study of the geometry of the points on the unit circle is called *circular trigonometry*.

Instead of looking at a point P on the unit circle $x^2 + y^2 = 1$, let's trace out the path of a point Q on the unit hyperbola $x^2 - y^2 = 1$. We will now define *hyperbolic* variants of the sin and cos functions to describe the coordinates of the point Q. This is called *hyperbolic trigonometry*. Doesn't that sound awesome? Next time your friends ask what you have been up to, tell them you are learning about hyperbolic trigonometry.

[An in-depth discussion on the conic sections]
http://astrowww.phys.uvic.ca/~tatum/celmechs/celm2.pdf

1.22 Solving systems of linear equations

We know that solving equations with one unknown—like $2x+4=7x$, for instance—requires manipulating both sides of the equation until the unknown variable is *isolated* on one side. For this instance, we can subtract $2x$ from both sides of the equation to obtain $4=5x$, which simplifies to $x=\frac{4}{5}$.

What about the case when you are given two equations and must solve for two unknowns? For example,

$$x+2y=5,$$
$$3x+9y=21.$$

Can you find values of x and y that satisfy both equations?

Concepts

- x, y: the two unknowns in the equations
- $eq1, eq2$: a system of two equations that must be solved *simultaneously*. These equations will look like

$$a_1x+b_1y=c_1,$$
$$a_2x+b_2y=c_2,$$

  where as, bs, and cs are given constants.

Principles

If you have n equations and n unknowns, you can solve the equations simultaneously and find the values of the unknowns. There are several different approaches for solving equations simultaneously. We'll learn about three ~~of these~~ different approaches in this section.

Solution techniques

~~Solving by equating~~

~~We want to solve the following system of equations: We can isolate x in both equationsby moving all other variables and constants to the right sides of the equations: Though the variable x is still unknown, we know two facts about it: x is equal to $5-2y$, and x is equal to $7-3y$. Therefore, We can solve for y by adding $3y$ to both sides and~~

~~subtracting 5 from both sides. We find $y = 2$~~When solving for two unknowns in two equations, the best approach is to *eliminate* one of the variables from the equations. By combining the two equations appropriately, we can reduce the problem to finding one unknown in one equation.

~~We found $y = 2$, but what is x? Easy. Plug the value $y = 2$ into any of the equations we started from. Let's try the first one : We're done, and $x = 1, y = 2$ is our solution.~~

~~Substitution~~Solving by substitution

~~Let's return to our set of equationsto see another approach for solving~~We want to solve the following system of equations:

$$\begin{aligned} x + 2y &= 5, \\ 3x + 9y &= 21. \end{aligned}$$

We can isolate x in the first equation to obtain

$$\begin{aligned} x &= 5 - 2y, \\ 3x + 9y &= 21. \end{aligned}$$

Now *substitute* the expression for x from the top equation into the bottom equation:

$$3(5 - 2y) + 9y = 21.$$

We just eliminated one of the unknowns by substitution. Continuing, we expand the bracket to find

$$15 - 6y + 9y = 21,$$

or

$$3y = 6.$$

~~Thus, we~~ We find $y = 2$, but what is x? Easy. To solve for x, ~~use the original~~ plug the value $y = 2$ into any of the equations we started from. Using the equation $x = 5 - 2y$~~to find $x = (5 - 2(2)) = 1$~~, we find $x = 5 - 2(2) = 1$.

~~Subtraction~~Solving by subtraction

~~There is a third way to solve the equations~~ Let's return to our set of equations to see another approach for solving:

$$\begin{aligned} x + 2y &= 5, \\ 3x + 9y &= 21. \end{aligned}$$

Observe that any equation will remain true if we multiply the whole equation by some constant. For example, we can multiply the first equation by 3 to obtain an equivalent set of equations:

$$3x + 6y = 15,$$
$$3x + 9y = 21.$$

Why did I pick 3 as the multiplier? By choosing this constant, the x terms in both equations now have the same coefficient.

Subtracting two true equations yields another true equation. Let's subtract the top equation from the bottom one:

$$\cancel{3x} - \cancel{3x} + 9y - 6y = 21 - 15 \quad \Rightarrow \quad 3y = 6.$$

The $3x$ terms cancel. This subtraction ~~became possible~~ eliminates the variable x because we multiplied the first equation by ~~3. We see that~~ 3. We find $y = 2$. ~~We can then substitute 2 for y in~~ To find x, substitute $y = 2$ into one of the original equations:

$$x + 2(2) = 5,$$

from which we deduce that $x = 1$.

Solving by equating

There is a third way to solve the equations:

$$x + 2y = 5,$$
$$3x + 9y = 21.$$

We can isolate x in both equations by moving all other variables and constants to the right sides of the equations:

$$x = 5 - 2y,$$
$$x = \frac{1}{3}(21 - 9y) = 7 - 3y.$$

Though the variable x is still unknown, we know two facts about it: x is equal to $5 - 2y$ and x is equal to $7 - 3y$. Therefore, we can eliminate x by equating the right sides of the equations:

$$5 - 2y = 7 - 3y.$$

We solve for y by adding $3y$ to both sides and subtracting 5 from both sides. We find $y = 2$. Plugging $y = 2$ into the equation $x = 5 - 2y$ we find

$$x = 5 - 2y = 5 - 2(2) = 1.$$

The solutions are $x = 1$ and $y = 2$.

Discussion

~~These techniques—elimination, substitution, and subtraction—can~~ The three elimination techniques—substitution, subtraction, and elimination by equating—can be extended to solve equations with more unknowns. There is actually an entire course called linear algebra, in which you~~will~~ 'll develop a more advanced, systematic approach for solving systems of linear equations.

1.23 Compound interest

Soon after ancient civilizations invented the notion of numbers, they started computing *interest* on loans. It is a good idea to know how interest calculations work so that you will be able to make informed decisions about your finances.

Percentages

We often talk about ratios between quantities, rather than mentioning the quantities themselves. For example, we can imagine average Joe, who invests \$1000 in the stock market and loses \$300 because the boys on Wall Street keep pulling dirty tricks on him. To put the number \$300 into perspective, we can say Joe lost 0.3 of his investment, or alternately, 30% of his investment.

To express a ratio as a percentage, multiply it by 100. The ratio of Joe's loss to investment is

$$R = 300/1000 = 0.3.$$

The same ratio expressed as a percentage gives

$$R = 300/1000 \times 100 = 30\%.$$

To convert from a percentage to a ratio, divide the percentage by 100.

Interest rates

Say you take out a \$1000 loan with an interest rate of 6% compounded annually. How much will you owe in interest at the end of the year?

Since 6% corresponds to a ratio of 6/100, and since you borrowed \$1000, the accumulated interest at the end of the year will be

$$I_1 = \frac{6}{100} \times \$1000 = \$60.$$

Compounding infinitely often

What is the effective APR if the nominal APR is 6% and the bank performs the compounding n times per year?

The annual growth ratio will be

$$\left(1+\frac{6}{100n}\right)^n,$$

where the interest rate per compounding period is $\frac{6}{n}$%, and there are n periods per year.

Consider a scenario in which the compounding is performed infinitely often. This corresponds to the case when the number n in the above equation tends to infinity (denoted $n \to \infty$). This is not a practical question, but it is an interesting avenue to explore nevertheless because it leads to the definition of the natural exponential function $f(x) = e^x$.

When we set $n \to \infty$ in the above expression, the annual growth ratio will be described by the exponential function base e as follows:

$$\lim_{n\to\infty}\left(1+\frac{6}{100n}\right)^n = \exp\!\left(\frac{6}{100}\right) = 1.0618365.$$

The expression "$\lim_{n\to\infty}$" is to be read as "in the limit when n tends to infinity." We will learn more about limits in Chapter 5.

A nominal APR of 6% with compounding that occurs infinitely often has an eAPR = 6.183%. After six years you will owe

$$L_6 = \exp\!\left(\frac{6}{100}\right)^6 \times 1000 = \$1433.33.$$

As you can see, the APR stays at a steady 6%—yet, the more frequent the compounding schedule, the more money you'll owe at the end of six years.

Links

[Very good article on interest calculations]
http://plus.maths.org/content/have-we-caught-your-interest

1.24 Set notation

A *set* is the mathematically precise notion for describing a group of objects. You ~~need not~~ don't need to know about sets to perform simple math; but more advanced topics require an understanding of what sets are ~~, as well as~~ and how to denote set membership and subset relations between sets.

Definitions

- *set*: a collection of mathematical objects ~~. The collection's contents are precisely defined.~~
- S, T: the usual variable names for sets
- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$: some important sets of numbers: the naturals, the integers, the rationals, and the real numbers, respectively.
- { definition }: the curly brackets surround the definition of a set, and the expression inside the curly brackets describes what the set contains.

Set operations:

- $S \cup T$: the *union* of two sets. The union of S and T corresponds to the elements in either S or T.
- $S \cap T$: the *intersection* of the two sets. The intersection of S and T corresponds to the elements in both S and T.
- $S \setminus T$: *set minus*. The difference $S \setminus T$ corresponds to the elements of S that are not in T.

Set relations:

- $\subset$: is a subset of
- $\subseteq$: is a subset of or equal to

Special mathematical shorthand symbols and their corresponding meanings:

- $\forall$: for all
- $\exists$: there exists
- $\nexists$: there doesn't exist
- $|$: such that
- $\in$: element of
- $\notin$: not an element of

Sets

Much of math's power comes from *abstraction*: the ability to see the bigger picture and think *meta* thoughts about the common relationships between math objects. We can think of individual numbers like 3, -5, and π, or we can talk about the *set* of *all* numbers.

It is often useful to restrict our attention to a specific *subset* of the numbers as in the following examples.

The broader class of real numbers also includes all rationals as well as irrational numbers like $\sqrt{2}$ and π:

$$\mathbb{R} \equiv \{\pi, e, -1.53929411\ldots,\ 4.99401940129401\ldots,\ \ldots\}.$$

Finally, we have the set of complex numbers:

$$\mathbb{C} \equiv \{1, i, 1+i, 2+3i, \ldots\}.$$

Note that the definitions of $\mathbb{R}$ and $\mathbb{C}$ are not very precise. Rather than giving a precise definition of each set inside the curly brackets as we did for $\mathbb{Z}$ and $\mathbb{Q}$, we instead stated some examples of the elements in the set. Mathematicians sometimes do this and expect you to guess the general pattern for all the elements in the set.

The following inclusion relationship holds for the fundamental sets of numbers:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}.$$

This relationship means every natural number is also an integer. Every integer is a rational number. Every rational number is a real. Every real number is also a complex number.

New vocabulary

The specialized notation used by mathematicians can be difficult to get used to. You must learn how to read symbols like $\exists$, $\subset$, $|$, and $\in$ and translate their meaning in the sentence. Indeed, learning advanced mathematics notation is akin to learning a new language.

To help you practice the new vocabulary, we will look at an ancient mathematical proof and express it in terms of modern mathematical symbols.

Square-root of 2 is irrational

Claim: $\sqrt{2} \notin \mathbb{Q}$. This means there do not exist numbers $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$ such that $m/n = \sqrt{2}$. The ~~same sentence~~ last sentence expressed in mathematical notation would read,

$$\nexists\, m \in \mathbb{Z}, n \in \mathbb{Z} \ \big|\ m/n = \sqrt{2}.$$

To prove ~~this claim wewill~~ the claim we'll use a technique called *proof by contradiction*. We ~~'ll~~ begin by assuming the opposite of what we want to prove: that there exist numbers $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$ such that $m/n = \sqrt{2}$. We'll then carry out some simple algebra steps and in the end we'll obtain an equation that is not true—we'll arrive at a contradiction. Arriving at a contradiction means our original supposition is wrong: there are no numbers $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$ such that $m/n = \sqrt{2}$.

Proof: Suppose there exist numbers $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$ such that $m/n = \sqrt{2}$. We can assume the integers m and n have no common factors. In particular, m and n cannot both be even, otherwise they would both contain at least one factor of 2. Next, we'll investigate whether m is an even number $m \in E$, or an odd number $m \in O$. Look back to Example 2 for the definitions of the sets O and E.

Before we check for even and oddness, it will help to point out the fact that the action of squaring an integer preserves its odd/even nature. An even number times an even number gives an even number: if $e \in E$ then $e^2 \in E$. Similarly, an odd number times an odd number gives an odd number: if $o \in O$ then $o^2 \in O$.

We proceed with the proof. We assume $m/n = \sqrt{2}$. Taking the square of both sides of this equation, we obtain

$$\frac{m^2}{n^2} = 2 \qquad \Rightarrow \qquad m^2 = 2n^2.$$

~~Let's analyze this~~ If we analyze the last equation in more detail, we can conclude that m cannot be an odd number, or written "$m \notin O$" in math. If m is an odd number then m^2 will also be odd, but this would contradict the above equation since the right-hand side of the equation ~~contains a factor of 2. Recall that any~~ contains the factor 2 and every number containing a factor 2 is even, ~~so if~~ not odd. If m is an integer ($m \in \mathbb{Z}$) and m is not odd ($m \notin O$) then it must be that ~~both sides of the equation are even : $m^2 \in E$ and so~~ m is even ($m \in E$~~.~~).

If m is even, then it ~~must contain~~ contains a factor of 2, so it can be written as $m = 2q$ where q is some other number $q \in \mathbb{Z}$. The exact value of q is not important. Let's revisit the equation $m^2 = 2n^2$ once more, this time substituting $m = 2q$ into the equation:

$$(2q)^2 = 2n^2 \quad \Rightarrow \quad 2q^2 = n^2.$$

By a similar reasoning as before, we can conclude n ~~must be an even number:~~ cannot be odd ($n \notin O$) so n must be even ($n \in E$). However, this statement contradicts our ~~previous statement~~ initial assumption that m and n ~~cannot both be even~~do not have any common factors!

The fact that we arrived at a contradiction means we must have made a mistake somewhere in our reasoning. Since each of the steps we carried out were correct, the mistake must be in the original premise, namely that "there exist numbers $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$ such that $m/n = \sqrt{2}$." Rather, the opposite must be true: "there do not exist numbers $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$ such that $m/n = \sqrt{2}$." The last ball is thrown in the positive y-direction with an initial velocity of $v_i = 12$[m/s]. The ball reaches a maximum height of $\max\{y(t)\} = \frac{12^2}{2\times 9.81} = 7.3$[m] at $t = 12/9.81 = 1.22$[s], then hits the ground after a total flight time of $t_f = 2\sqrt{\frac{2\times 7.3}{9.81}} = 2.44$[s]. The *measurement units* of physical quantities throughout this book are denoted in square brackets, like in the example above. Learning about the different measurement units is an important aspect of *physics vision*.

Why learn physics?

The main reason why you should learn physics is ~~to experience~~ for the *knowledge buzz*. You will learn how to calculate the motion of objects, predict the outcomes of collisions, describe oscillations, and many other useful things. As you develop your physics skills, you will be able to use physics equations to derive one physical quantity from another. For example, we can predict the maximum height reached by a ball, if we know its initial velocity when thrown. ~~Physics is a bit like playing LEGOswith a bunch of shiny new scientific building blocks.~~ The equations of physics are a lot like LEGOs; your job is to figure out different ways to connect them together.

By learning how to solve complicated physics problems, you will develop your analytical skills. Later on, you can apply these skills to other areas of life. Even if you don't go on to study science, the expertise you develop in solving physics problems will help you tackle complicated problems in general. As proof of this statement, consider the fact that companies like to hire physicists even for positions unrelated to physics: they feel confident that candidates who understand physics will be able to figure out all the business stuff easily.

Intro to science

Perhaps the most important reason you should learn physics is because it represents the golden standard for the scientific method. First of all, physics deals only with concrete things that can be **measured**. There are no feelings or subjectivities in physics. Physicists must derive mathematical models that **accurately describe** and **predict** the outcomes of experiments. Above all, we can **test** the validity of the physical models by running experiments and comparing the predicted outcome with what actually happens in the lab.

The key ingredient in scientific thinking is skepticism. Scientists must convince their peers that their equations are true without a doubt. The peers shouldn't need to *trust* the scientist; rather, they can carry out their own tests to see if the equation accurately predicts what happens in the real world. For example, let's say I claim that the height of a ball thrown up in the air with speed 12[m/s] is described by the equation $y_c(t) = \frac{1}{2}(-9.81)t^2 + 12t + 0$. To test whether this equation is true, you can perform a throwing-the-ball-in-the-air experiment and record the motion of the ball as a video. You can then compare the motion parameters observed in the video with those predicted by the claimed equation $y_c(t)$.

- **Maximum height reached** One thing you can check is whether the equation $y_c(t)$ predicts the ball's maximum height $y_{\max}$. The claimed equation predicts the ball will reach its maximum height at $t = 1.22$[s]. The maximum height predicted is $\max_t\{y_c(t)\} = y_c(1.22) = 7.3$[m]. You can compare this value with the maximum height $y_{\max}$ you observe in the video.
- **Total time of flight** You can also check whether the equation $y_c(t)$ correctly predicts the time when the ball will fall back to the ground. Using the video, suppose you measure the time it took the ball to fall back to the ground to be $t_{\text{fall}} = 2.44$[s]. If the equation $y_c(t)$ is correct, it should predict a height of zero metres for the time t_{fall}.

If both predictions of the equation $y_c(t)$ match your observations from the video, you can start to believe the claimed equation of motion $y_c(t)$ is truly an accurate model for the real world.

The scientific method depends on this interplay between experiment and theory. Theoreticians prove theorems and derive equations, while experimentalists test the validity of equations. The equations that accurately predict the laws of nature are kept while inaccurate models are rejected. At the same time, experimentalists constantly measure new data and challenge theoreticians to come up with equations that correctly describe the new measurements.

Equations of physics

The best physics equations are collected in textbooks. Physics textbooks contain only equations that have been extensively tested and are believed to be true. Good physics textbooks also explain how the equations are *derived* from first principles. This is important, because it is much easier to understand a few fundamental principles of physics, rather than memorize a long list of formulas. Understanding trumps memorization any day of the week.

The next section will teach you about three equations that fully describe the motion of any object: $x(t)$, $v(t)$, and $a(t)$. Using these equations and the equation-solving techniques from Chapter 1, we can predict pretty much anything we want about the position and velocity of objects undergoing *constant acceleration*.

Instead of ~~memorizing the~~ asking you to memorize these equations, I'll show you a cool trick for obtaining one equation of motion from another. These three equations describe different aspects of the same motion, so it's no surprise the equations are related. While you are not required to know how to derive the equations of physics, you do need to know how to use all these equations. Learning a bit of theory is a good deal: just a few pages of "difficult" theory (integrals) will give you a deep understanding of the relationship between $a(t)$, $v(t)$, and $x(t)$. This way, you can rely on your newly expanded math knowledge, rather than remember three separate formulas!

2.2 Kinematics

Kinematics (from the Greek word *kinema* for *motion*) is the study of trajectories of moving objects. The equations of kinematics can be used to calculate how long a ball thrown upward will stay in the air, or to calculate the acceleration needed to go from 0 to 100[km/h] in 5 seconds. To carry out these calculations, we need to choose the right *equation of motion* and figure out the values of the *initial conditions* (the initial position x_i and the initial velocity v_i). Afterward, we plug the known values into the appropriate equation of motion and solve for the unknown using one or two simple algebra steps. This entire section boils down to three equations and the plug-number-into-equation skill.

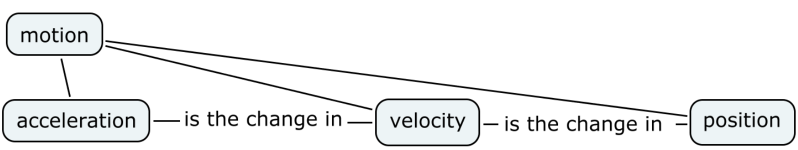


Figure 2.1: The motion of an object is described by its position, velocity, and acceleration functions.

This section is here to teach you how to use the equations of motion and help you understand the concepts of velocity and acceleration. You'll also learn how to recognize which equations to use when solving different types of physics problems.

Concepts

The key notions for describing the motion of objects are:

- t: the time. Time is measured in seconds [s].
- $x(t)$: an object's position as a function of time—also known as the equation of motion. Position is measured in metres [m] and depends on the time t.
- $v(t)$: the object's velocity as a function of time. Velocity is measured in metres per second [m/s].
- $a(t)$: the object's acceleration as a function of time. Acceleration is measured in metres per second squared [m/s^2].
- $x_i = x(0), v_i = v(0)$: the object's initial position and velocity, as measured at $t = 0$. Together x_i and v_i are known as the *initial conditions*.

Position, velocity, and acceleration

The motion of an object is characterized by three functions: the position function $x(t)$, the velocity function $v(t)$, and the acceleration function $a(t)$. The functions $x(t)$, $v(t)$, and $a(t)$ are connected—they all describe different aspects of the same motion.

You are already familiar with these notions from your experience of riding in a car. The equation of motion $x(t)$ describes the position of the car as a function of time. The velocity describes the change in the position of the car, or mathematically,

$$v(t) \equiv \text{rate of change in } x(t).$$

If we measure x in metres [m] and time t in seconds [s], then the units of $v(t)$ will be metres per second [m/s]. For example, an object moving with at a constant ~~speed of 30~~velocity of +30[m/s] will ~~change~~ increase it's position by 30[m] each second. Note that the velocity $v(t)$ could be positive or negative. The *speed* of an object is defined as the absolute value of it's velocity $|v(t)|$.

The rate of change of an object's velocity is called *acceleration*:

$$a(t) \equiv \text{rate of change in } v(t).$$

Acceleration is measured in metres per second squared [m/s^2]. A constant positive acceleration means the velocity of the motion is steadily increasing, similar to pressing the gas pedal. A constant negative acceleration means the velocity is steadily decreasing, similar to pressing the brake pedal.

In a couple of paragraphs, we'll discuss the exact mathematical equations for $x(t)$, $v(t)$, and $a(t)$, but before we dig into the math, let's look at the example of the motion of a car illustrated in Figure 2.2.

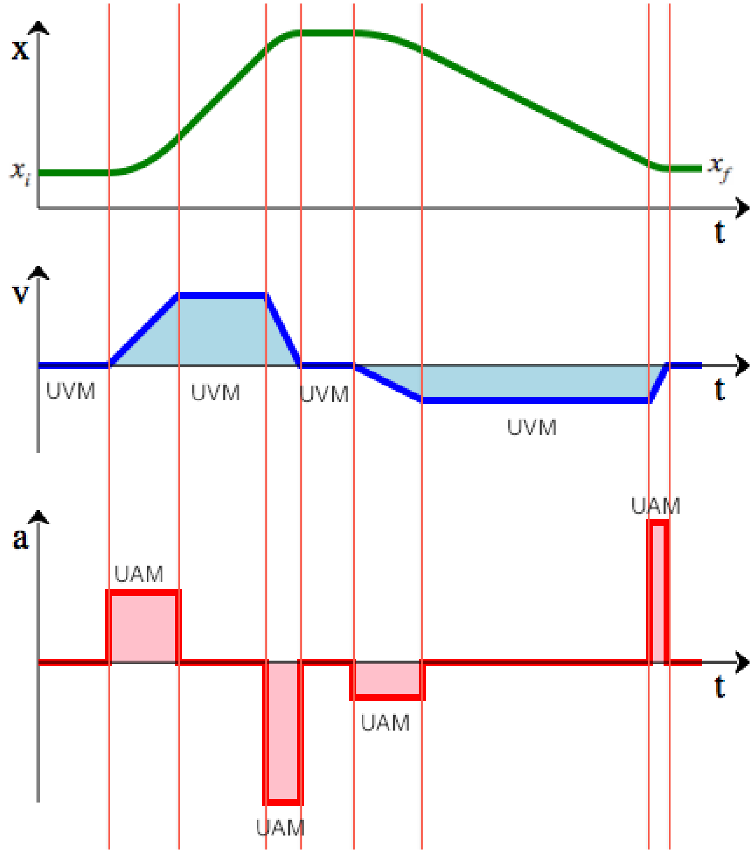


Figure 2.2: The illustration shows the simultaneous graphs of the position, velocity, and acceleration of a car during some time interval. The car starts from an initial position x_i where it sits still for some time. The driver then floors the pedal to produce a maximum acceleration for some time, and the car picks up speed. The driver then ~~releases~~ eases off the accelerator, keeping it pressed enough to maintain a constant speed. Suddenly the driver sees a police vehicle in the distance and slams on the brakes (negative acceleration) and shortly afterward brings the car to a stop. The driver waits for a few seconds to make sure the cops have passed. Next, the driver switches into reverse gear and adds gas. The car accelerates backward for a bit, then maintains a constant backward speed for an extended period of time. Note how "moving backward" corresponds to negative velocity. In the end the driver slams on the brakes again to stop the car. Notice that braking corresponds to positive acceleration when the motion is in the negative direction. The car's final position is x_f.

We can observe two distinct types of motion in the situation described in Figure 2.2. During some times, the car undergoes motion at a constant velocity (uniform velocity motion, UVM). During other times, the car undergoes ~~motions~~ motion with constant acceleration (uniform acceleration motion, UAM). There exist many other types of motion, but for the purpose of this section we'll focus on these two types of motion.

- UVM: During times when there is no acceleration, the car maintains a uniform velocity and therefore $v(t)$ is a constant function. For motion with constant velocity, the position function is a line with a constant slope because, by definition, $v(t) =$ slope of $x(t)$.
- UAM: During times where the car experiences a constant acceleration $a(t) = a$, the velocity of the function changes at a constant rate. The rate of change of the velocity is constant $a =$ slope of $v(t)$, so the velocity function looks like a line with slope a. The position function $x(t)$ has a curved shape (quadratic) during moments of constant acceleration.

Formulas

There are basically four equations you need to know for this entire section. Together, these four equations fully describe all aspects of motion with constant acceleration.

Uniformly accelerated motion (UAM)

If the object undergoes a *constant* acceleration $a(t) = a$—like a car when you floor the *accelerator*—then its motion can be described by the following equations:

$$a(t) = a, \tag{2.1}$$
$$v(t) = at + v_i, \tag{2.2}$$
$$x(t) = \tfrac{1}{2}at^2 + v_i t + x_i, \tag{2.3}$$

where v_i is the initial velocity of the object and x_i is its initial position.

Here is another useful equation to remember:

$$[v(t)]^2 = v_i^2 + 2a[x(t) - x_i],$$

which is usually written

$$v_f^2 = v_i^2 + 2a\Delta x, \tag{2.4}$$

where v_f denotes the final velocity (at $t = t_f$) and Δx denotes the *change* in the x-coordinate between $t = 0$ and $t = t_f$. The triangle thing Δ is the capital Greek letter *delta*, which is often used to denote the change in quantities. Using ~~this notation~~the Δ-notation, we can rewrite ~~Formula~~ equation (2.2) ~~in Δ-notation as~~ as follows: $\Delta v = a\Delta t$, where $\Delta v \equiv v_f - v_i$ and $\Delta t \equiv t_f - t_i$.

That~~is it. Memorize these~~ 's it! Memorize these four equations, plug-in the right numbers, and you can solve any kinematics problem humanly imaginable.

Uniform velocity motion (UVM)

The special case where there is zero acceleration ($a = 0$), is called *uniform velocity motion* or UVM. The velocity stays uniform (constant) because there is no acceleration. The following three equations describe the motion of an object with uniform velocity:

$$\begin{aligned} a(t) &= 0, \\ v(t) &= v_i, \\ x(t) &= v_i t + x_i. \end{aligned}$$

As you can see, these are really the same equations as in the UAM case above, but because $a = 0$, some terms are missing.

Free fall

We say an object is in *free fall* if the only force acting on it is the force of gravity. On the surface of the Earth, the force of gravity produces a constant acceleration of $a_y = -9.81[\text{m/s}^2]$. The negative sign is there because the gravitational acceleration is directed downward, and we assume the y-axis points upward. Since the gravitational acceleration is constant, we can use the UAM equations to find the height $y(t)$ and velocity $v(t)$ of objects in free fall.

Examples

Now we'll illustrate how the equations of kinematics are used.

Moroccan example Suppose your friend wants to send you a ball wrapped in aluminum foil by dropping it from his balcony, which is located at a height of $y_i = 44.145[\text{m}]$. How long will it take for the ball to hit the ground?

Imagine the apartment building as a y-axis that measures distance upward starting from the ground floor. We know the balcony is located at a height of $y_i = 44.145$[m], and that at $t = 0$[s] the ball starts with $v_i = -10$[m/s]. The initial velocity is negative because it points in the opposite direction of the y-axis. We also know there is an acceleration due to gravity of $a_y = -9.81$[m/s^2].
We start by writing the general UAM equation:

$$y(t) = \tfrac{1}{2}a_y t^2 + v_i t + y_i.$$

To find the time when the ball will hit the ground, we must solve for t in the equation $y(t) = 0$. Plug all the known values into the UAM equation,

$$y(t) = 0 = \tfrac{1}{2}(-9.81)t^2 - 10t + 44.145,$$

and solve for t using the quadratic formula. First, rewrite the quadratic equation in standard form:

$$0 = \underbrace{4.905}_{a} t^2 + \underbrace{10.0}_{b} t \quad \underbrace{-44.145}_{c}.$$

Then solve using the quadratic equation:

$$t_{\text{fall}} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-10 \pm \sqrt{100 + 866.12}}{9.81} = 2.15 \qquad [\text{s}].$$

We ignore the negative-time solution because it corresponds to a time in the past. Compared to the first Moroccan example, we see that throwing the ball downward makes it fall to the ground faster.

Discussion

Most kinematics problems you'll be ~~asked to solve will~~ solving follow the same pattern as the examples above. Given some initial values, you~~will~~ 'll be asked to solve for some unknown quantity.

It~~is~~ 's important to keep in mind the *signs* of the numbers you plug into the equations. You should always draw the coordinate system and indicate clearly (to yourself) the x-axis, which measures the object's displacement. A velocity or acceleration quantity that points in the same direction as the x-axis is a positive number, while quantities pointing in the opposite direction are negative numbers.

By the way, all this talk about $v(t)$ being the "rate of change of $x(t)$" is starting to get on my nerves. The expression "rate of change of" is an indirect way of saying the calculus term *derivative*. In order to use this more precise terminology throughout the remainder of the book, we~~will~~ 'll now take a short excursion into the land of calculus to define two fundamental concepts: derivatives and integrals.

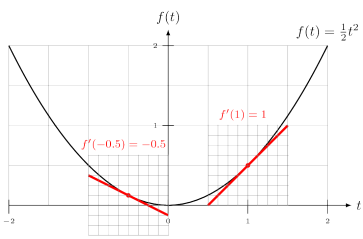
2.3 Introduction to calculus

Calculus is the study of functions and their properties. The two operations in the study of calculus are derivatives—which describe how quantities *change over time*—and integrals, which are used to calculate the total amount of a quantity *accumulated* over a time period.

Derivatives

The derivative function $f'(t)$ describes how the function $f(t)$ changes over time. The derivative encodes the information about the instantaneous *rate of change* of the function $f(t)$, which is the same as the *slope* of the ~~function $f(t)$~~graph of the function at that point:

$$f'(t) \equiv \text{slope}_f(t) = \frac{\text{change in } f(t)}{\text{change in } t} = \frac{f(t+\Delta t) - f(t)}{\Delta t}.$$

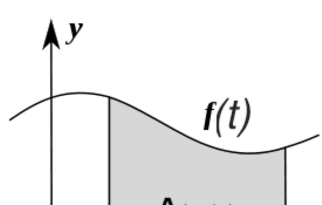


If the derivative $f'(t)$ is equal to 5 units per second, this means that $f(t)$ changes by 5 units each second. The derivative of the constant function is zero because it has zero rise over run everywhere. The derivative of the function $f(t) = mt + b$ (a line) is the constant function $f'(t) = m$. More generally, the instantaneous slope of a function is different for different values of t, as illustrated in the figure.

The derivative operation is denoted by several names and symbols: $Df(t) = f'(t) = \frac{df}{dt} = \frac{d}{dt}\{f(t)\} = \dot{f}$ ~~, all of which~~ and all these symbols carry the same meaning. Think of $f'(t)$ not as a separate entity from $f(t)$, but as a *property* of the function $f(t)$. It's best to think of the derivative as an ~~operator~~ *operator* $\frac{d}{dt}$ that you can apply to any function to obtain its slope information. ~~Derivatives are used in many areas of science.~~

Integrals

An integral corresponds to the computation of the *area* enclosed between the curve $f(t)$ and the x-axis over some in-



terval:

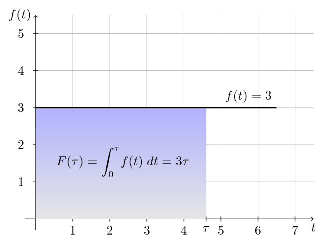
$$A(a, b) \equiv \int_{t=a}^{t=b} f(t)\, dt.$$

The symbol $\int$ is shorthand for *sum*. Indeed, the area under the curve corresponds to the sum of the values of the function $f(t)$ between $t = a$ and $t = b$.

The integral is the total of f between a and b.

Example 1

We can easily find the area under the ~~constant function $f(t) = 3$~~ graph of the constant function $f(t) = 3$ between any two points because the region under the curve is rectangular. ~~We choose to use~~ Choosing $t = 0$ as the ~~reference pointand compute the integral~~ starting point, we obtain the integral function $F(\tau)$, which corresponds to the area under $f(t)$ ~~starting from~~ between $t = 0$ and ~~going until~~ $t = \tau$:

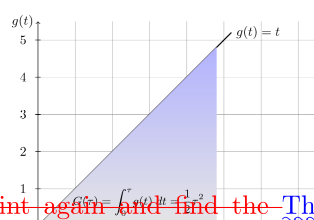


$$F(\tau) \equiv A(0, \tau) = \int_0^\tau f(t)\, dt = 3\tau.$$

The area is equal to the rectangle's height times its width.

Example 2

Consider now the area under the graph of the line $g(t) = t$, starting from $t = 0$. Since the region under the curve is triangular, we can compute its area. Recall the area of a triangle is given by the length of its base times its height divided by 2.



~~We choose $t = 0$ as our starting point again and find the~~ The general formula for the area ~~below~~ under $g(t)$ from $t = 0$ until $t = \tau$ is described by the following integral calculation:

$$G(\tau) \equiv A(0, \tau) = \int_0^\tau g(t)\, dt = \frac{\tau \times \tau}{2} = \tfrac{1}{2}\tau^2.$$

We~~are~~ 're able to compute the above integrals thanks to the simple ~~geometry~~ geometries of the areas under the ~~curves~~graphs. Later in this book (Chapter 5), we'll develop techniques for finding integrals (areas under the curve) of more complicated functions. In fact, there is an entire course called ~~Integral Calculus~~integral calculus, which is dedicated to the task of finding integrals.

~~What I need you to remember for~~ But don't worry, you don't need to know *everything* about integrals to learn physics. What is important right now is that ~~the~~ you understand the concept of integration. The integral of a function ~~gives you~~ is the area under the ~~curve~~graph of the function, which is in some sense the total amount of the function accumulated during ~~that period.~~ ~~You should also remember the following two formulas :~~ some interval of time. For the most part of the first-year physics, the only integral formulas you'll need to know are

$$\int_0^\tau a\, dt = a\tau \qquad \text{and} \qquad \int_0^\tau at\, dt = \tfrac{1}{2}a\tau^2.$$

The first integral describes the general calculation of the area under a constant function, like in Example 1. The second formula is a generalization of the formula we ~~saw in Example~~derived in Example 2. ~~Using the above~~ Using these formulas in combination, you can ~~now~~ compute the integral ~~under~~ of an arbitrary line $h(t) = mt + b$ as follows:

$$H(\tau) = \int_0^\tau h(t)\, dt = \int_0^\tau (mt+b)\, dt = \int_0^\tau mt\, dt + \int_0^\tau b\, dt = \tfrac{1}{2}m\tau^2 + b\tau.$$

~~Do we really need integrals? How often will you need~~

Regroup

At this point you're probably on the fence about the new calculus concepts. On the one hand, calculating slopes (derivatives) and areas under the curve (integrals) seem trivial tasks. On the other hand, seeing five different notations for the derivative and the weird integral sign has probably put some fear in you. You might be wondering whether you *really* need to learn about derivatives and integrals. How often do you have to compute the area ~~below a function $f(t)$~~ under the graph of a function in the real world? It turns out that "calculating the area under a curve" is ~~a very useful operation because it allows us to undo the derivative operation. Understanding the relationship between the derivativeand the integral operations will allow you to solve many problems.~~ very useful since it is the "undo operation" for the derivative.

Inverse operations

The integral is the inverse operation of the derivative. Many equations in math and physics involve the *derivative* of some unknown function. Understanding the inverse relationship between integrals and derivatives will allow you to solve for the unknown function in these equations.

You should already be familiar with the inverse relationship between functions. When solving equations (page 4), we use inverse functions to *undo* functions that stand in our way as we try to isolate the unknown x. Similarly, we use the integral operation to *undo* the effects of the derivative operation when we try to solve for some unknown function $f(t)$. For example, suppose $g(t)$ is a known function and we're trying to solve for $f(t)$ in the equation

$$\frac{d}{dt}\{f(t)\} = g(t).$$

Taking the integral on the left-hand side of the equation will undo the derivative operation. To keep the equality true, we must apply the integration operation on both sides of the equation to obtain

$$\begin{aligned}\int \frac{d}{dt}\{f(t)\}\,dt &= \int g(t)\,dt,\\ f(t) &= \int g(t)\,dt.\end{aligned}$$

Calculating the total of the instantaneous changes in f is the same as the final change in f. Every time you want to *undo* a derivative, you can apply the integral operation~~.~~ ~~There~~, however, there is a little technical ~~complication we must discuss.~~ detail that we must clarify to make this statement precise.

The integral isn't *exactly* the inverse of the derivative—there ~~exists a tricky dependence on the limits of integration.~~ is a tricky extra constant factor that appears when we integrate. Let's analyze in more detail what happens when we perform the combo of the derivative operation followed by the integral operation on some function $f(t)$. Suppose we are given the derivative function $f'(t)$ and asked to integrate it between $t = 0$ and $t = \tau$. Intuitively, this integral corresponds to calculating the **total of the changes** in $f(t)$ during that time interval. Recall the notation for "change in f" $\Delta f \equiv f(\tau) - f(0)$, which we used previously. This notation makes it easy to see how the integral over $f'(t)$ corresponds to the total change in $f(t)$ between $t = 0$ and $t = \tau$:

$$\int_0^\tau f'(t)\,dt = \Delta f = f(\tau) - f(0).$$

Rewriting this equation to isolate $f(\tau)$ we obtain ~~The answer~~

$$f(\tau) = f(0) \quad + \int_0^\tau f'(t)\,dt.$$

Note that the expression for $f(\tau)$ depends on the value of $f(t)$ at $t = 0$, ~~the initial conditions.~~ which we call the *initial value* of the function. In physics problems, the initial values of the equations of motion $x(0) \equiv x_i$ and $v(0) \equiv v_i$ are called the *initial conditions*.

Banking example To illustrate how derivative and integral operations apply to the real world, I'll draw an analogy from a scenario that every student is familiar with. Consider the function $\mathrm{ba}(t)$, which represents your bank account balance at time t. Also consider the function $\mathrm{tr}(t)$, which corresponds to the transactions (deposits and withdrawals) on your account.

The function $\mathrm{tr}(t)$ is the derivative of the function $\mathrm{ba}(t)$. If you ask, "how does my balance change over time?" the answer is the function $\mathrm{tr}(t)$. Using mathematical symbols, we can represent this relationship as

$$\mathrm{tr}(t) = \frac{d}{dt}\left\{\mathrm{ba}(t)\right\}.$$

If the derivative is positive, your account balance is growing. If the derivative is negative, your account balance is depleting.

Suppose you have a record of all the transactions on your account $\mathrm{tr}(t)$, and you want to compute the final account balance at the end of the month. Since $\mathrm{tr}(t)$ is the derivative of $\mathrm{ba}(t)$, you can use an integral (the inverse operation of the derivative) to obtain $\mathrm{ba}(t)$. Knowing the balance of your account at the beginning of the month, you can predict the balance at the end of the month by calculating the following integral:

$$\mathrm{ba}(30) = \mathrm{ba}(0) + \int_0^{30} \mathrm{tr}(t)\,dt.$$

This calculation makes sense since $\mathrm{tr}(t)$ represents the instantaneous changes in $\mathrm{ba}(t)$. If you want to find the overall change from day 0 until day 30, you can compute the total of all the changes in the account balance.

We use integrals every time we need to calculate the total of some quantity over a time period. In the next section, we'll see how these integration techniques can be applied to the subject of kinematics, and how the equations of motion for UAM are derived from first principles.

2.4 Kinematics with calculus

To carry out kinematics calculations, all we need to do is plug the initial conditions (x_i and v_i) into the correct equation of motion. But how did Newton come up with these equations in the first place? Now that you know Newton's mathematical techniques (calculus), you can see for yourself how the equations of motion are derived.

Concepts

Recall the kinematics concepts related to the motion of objects:

- t: time
- $x(t)$: position as a function of time
- $v(t)$: velocity as a function of time
- $a(t)$: acceleration as a function of time
- $x_i = x(0), v_i = v(0)$: the initial conditions

Position, velocity, and acceleration revisited

The equations of kinematics are used to predict the motion of objects. Suppose you know the acceleration of the object $a(t)$ at all times t. Can you find $x(t)$ starting from $a(t)$?

The equations of motion $x(t)$, $v(t)$, and $a(t)$ are related:

$$a(t) \xleftarrow{\frac{d}{dt}} v(t) \xleftarrow{\frac{d}{dt}} x(t).$$

The velocity function is the derivative of the position function and the acceleration function is the derivative of the velocity function.

General procedure

If you know the acceleration of an object as a function of time $a(t)$, and you know its initial velocity $v_i = v(0)$, you can find its velocity function $v(t)$ for all later times using integration. This is because the acceleration function $a(t)$ describes the change in the object's velocity. If you know the object started with an initial velocity of $v_i \equiv v(0)$, the velocity at a later time $t = \tau$ is equal to v_i plus the total acceleration of the object between $t = 0$ and $t = \tau$:

$$v(\tau) = v_i + \int_0^{\tau} a(t)\, dt.$$

If you know the initial position x_i and the velocity function $v(t)$, you can find the position function $x(t)$ by using integration. We find the position at time $t = \tau$ by adding all the velocities (the changes in the object's position) that occurred between $t = 0$ and $t = \tau$:

$$x(\tau) = x_i + \int_0^\tau v(t)\, dt.$$

The procedure for finding $x(t)$ starting from $a(t)$ can be summarized as follows:

$$a(t) \quad \xrightarrow{v_i + \int dt} \quad v(t) \quad \xrightarrow{x_i + \int dt} \quad x(t).$$

Next, I'll illustrate how you can apply this procedure to the important special case of an object undergoing uniformly accelerated motion.

Derivation of the UAM equations of motion

Consider an object undergoing uniformly accelerated motion (UAM) with acceleration function $a(t) = a$. Suppose we know the initial velocity of $v_i \equiv v(0)$, and we want to find the velocity at a later time $t = \tau$. We compute $v(\tau)$ using the following integral:

$$v(\tau) = v_i + \int_0^\tau a(t)\ dt = v_i + \int_0^\tau a\ dt = v_i + a\tau.$$

Velocity as a function of time is given by the initial velocity v_i added to the integral of the acceleration. The integration ~~step~~ can be visualized as the calculation of the area of a rectangle, similar to the calculation we saw in Example 1 on page 112.

You can also use integration to find the position function $x(t)$ if you know the initial position x_i and the velocity function $v(t)$. The formula is

$$x(\tau) = x_i + \int_0^\tau v(t)\ dt = x_i + \int_0^\tau (v_i + at)\ dt = x_i + v_i\tau + \tfrac{1}{2}a\tau^2.$$

The integration step can be visualized as the calculation of the area of a triangle with slope a stacked on top of a rectangle of height v_i.

Note that the above calculations required knowing the initial conditions x_i and v_i. These initial values were required because the integral calculations we performed only told us the *change* in the quantities relative to their initial values.

The fourth equation

We can derive the fourth equation of motion,

$$v_f^2 = v_i^2 + 2a(x_f - x_i),$$

by combining the equations of motion $v(t)$ and $x(t)$. Let's see how. Start by squaring both sides of the velocity equation $v_f = v_i + at$ to obtain

$$v_f^2 = (v_i + at)^2 = v_i^2 + 2av_it + a^2t^2 = v_i^2 + 2a[v_it + \tfrac{1}{2}at^2].$$

The term in the square bracket is equal to $\Delta x = x(t) - x_i = x_f - x_i$.

Applications of derivatives

Recall that the velocity and the acceleration functions are obtained by taking derivatives of the position function:

$$x(t) \xrightarrow{\frac{d}{dt}} v(t) \xrightarrow{\frac{d}{dt}} a(t).$$

We just saw how to use integration to follow this chain of operations in reverse to obtain $x(t)$ for the special case of constant acceleration:

$$\begin{aligned} a(t) &\equiv a, \\ v(t) &\equiv v_i + \int_0^t a(\tau)\, d\tau = v_i + at, \\ x(t) &\equiv x_i + \int_0^t v(\tau)\, d\tau = x_i + v_it + \tfrac{1}{2}at^2. \end{aligned}$$

Note that, in addition to the integral calculations, the formulas for $v(t)$ and $x(t)$ require some additional information—the initial value of the function.

Earlier we defined the derivative operator $\frac{d}{dt}$ that computes the derivative function $f'(t)$, which tells us the slope of the function $f(t)$. There are several derivative formulas that you need to learn to be proficient at calculus. We'll get to that in Chapter 5. For now, the only derivative formula that you'll need is the *power rule* for derivatives:

$$\text{if} \quad f(t) = At^n \quad \text{then} \quad f'(t) = nAt^{n-1}.$$

Using this formula on each term in the function $f(t) = A + Bt + Ct^2$ we find its derivative is $\frac{df}{dt} \equiv f'(t) = 0 + B + 2Ct$.

Let's now use the derivative to verify that the equations of motion we obtained above satisfy $x'(t) = v(t)$ and $v'(t) = a(t)$. Applying the derivative operation to both sides of the equations we obtain

$$\begin{aligned} a'(t) &\equiv 0, \\ v'(t) &\equiv \tfrac{d}{dt}\{v_i + at\} = \cancel{\tfrac{d}{dt}\{v_i\}} + \tfrac{d}{dt}\{at\} = 0 + a &&= a(t), \\ x'(t) &\equiv \cancel{\tfrac{d}{dt}\{x_i\}} + \tfrac{d}{dt}\{v_it\} + \tfrac{d}{dt}\{\tfrac{1}{2}at^2\} = 0 + v_i + at &&= v(t). \end{aligned}$$

Note that computing the derivative of a function kills the information about it's initial value; the derivative contains only information about the changes in $f(t)$.

Let's summarize what we learned up until now about derivatives and integrals. Integrals are useful because they allow us to compute $v(t)$ from $a(t)$, and $x(t)$ from $v(t)$. The derivative operation is useful because it allows us to obtain $v(t)$ if we know $x(t)$, and/or obtain $a(t)$ if we know $v(t)$. Recall that $x(t)$, $v(t)$, and $a(t)$ correspond to three different aspects of the same motion, as shown in Figure 2.2 on page 106. The operations of calculus allow us to move freely between the different descriptions of the motion.

Discussion

According to Newton's second law of motion, forces are the cause of acceleration and the formula that governs this relationship is

$$F_{\text{net}} = ma,$$

where F_{net} is the magnitude of the net force acting on the object.

In Chapter 4 we'll learn about *dynamics*, the study of the different kinds of forces that can act on objects: gravitational force $\vec{F}_g$, spring force $\vec{F}_s$, friction force $\vec{F}_f$, and other forces. To find an object's acceleration, we must add together all the forces acting on the object and divide by the object's mass:

$$\sum F_i = F_{\text{net}} \qquad \Rightarrow \qquad a = \frac{1}{m}F_{\text{net}}.$$

The physics procedure for predicting the motion of ~~objects~~ an object given the forces acting on it can be summarized as follows:

$$\underbrace{\frac{1}{m}\left(\sum \vec{F} = \vec{F}_{\text{net}}\right)}_{\text{dynamics}} = \underbrace{a(t) \overset{v_i+\int dt}{\longrightarrow} v(t) \overset{x_i+\int dt}{\longrightarrow} x(t)}_{\text{kinematics}}.$$

Free fall revisited

The force of gravity acting on an object of mass m on the surface of the Earth is given by $\vec{F}_g = -mg\hat{y}$, where $g = 9.81[\text{m/s}^2]$ is the *gravitational acceleration* on the surface of the Earth. We previously discussed that an object is in *free fall* when the only force acting on it is the force of gravity. In this case, Newton's second law tells us

$$\vec{F}_{\text{net}} = m\vec{a}$$
$$-mg\hat{y} = m\vec{a}.$$

Dividing both sides by the mass, we see the acceleration of an object in free fall is $\vec{a} = -9.81\hat{y}$.

It's interesting to note that an object's mass does not affect its acceleration during free fall. The force of gravity is proportional to the mass of the object, but acceleration is inversely proportional to the mass of the object; overall, it holds that $a_y = -g$ for objects in free fall, regardless of their mass. This observation was first made by Galileo in his famous Leaning Tower of Pisa experiment. Galileo dropped a wooden ball and a metal ball (same shape, different mass) from the Leaning Tower of Pisa, and observed that they fell to the ground at the same time. Search for "Apollo 15 feather and hammer drop" on `YouTube` to see this experiment performed on the Moon.

What next?

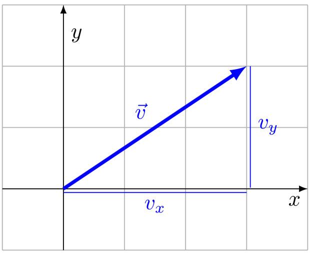
You might have noticed that in the last couple of paragraphs we started putting little arrows on top of certain quantities. The arrows are there to remind you that forces, velocities, and accelerations are *vector quantities*. Before we proceed with the physics lessons, we ~~'ll make an interesting~~ must make a short mathematical digression to introduce vectors.

Chapter 3

Vectors

In this chapter we~~will~~'ll learn how to manipulate multi-dimensional objects called vectors. Vectors are the precise way to describe directions in space. We need vectors in order to describe physical quantities like the velocity of an object, its acceleration, and the net force acting on the object.

Vectors are built from ordinary numbers, which form the *components* of the vector. You can think of a vector as a list of numbers, and *vector algebra* as operations performed on the numbers in the list. Vectors can also be manipulated as geometrical objects, represented by arrows in space. The arrow that corresponds to the vector $\vec{v} = (v_x, v_y)$ starts at the origin $(0, 0)$ and ends at the point (v_x, v_y). The word vector comes from the Latin *vehere*, which means *to carry*. Indeed, the vector $\vec{v}$ takes the point $(0, 0)$ and carries it to the point (v_x, v_y).



This chapter will introduce you to vectors, vector algebra, and vector operations, which are very useful for solving physics problems. What you'll learn here applies more broadly to problems in computer graphics, probability theory, machine learning, and other fields of science and mathematics. It's all about vectors these days, so you better get to know them.

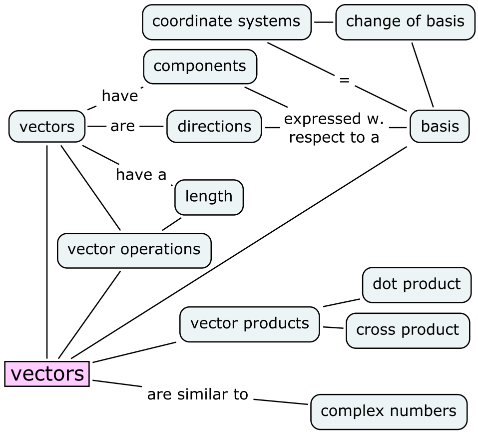


Figure 3.1: This figure illustrates the new concepts related to vectors. As you can see, there is quite a bit of new vocabulary to learn, but don't be ~~phased—all~~ fazed—all these terms are just fancy ways of talking about arrows.

3.1 Great outdoors

Vectors are directions for getting from point A to point B. Directions can be given in terms of street names and visual landmarks, or with respect to a coordinate system.

While on vacation in British Columbia, you want to visit a certain outdoor location your friend told you about. Your friend isn't available to take you there himself, but he has sent you *directions* for how to get to the place from the bus stop:

```
Sup G. Go to bus stop number 345. Bring a compass.
Walk 2 km north then 3 km east. You will find X there.
```

This text message contains all the information you need to find ~~*X*~~X.

Act 1: Following directions

You arrive at the bus station, located at the top of a hill. From this height you can see the whole valley, and along the hillside below spreads a beautiful field of tall crops. The crops are so tall they prevent anyone standing in them from seeing too far; good thing you have a compass. You align the compass needle so the red arrow points north. You walk 2 km north, then turn right (east) and walk another 3 km. You arrive at X.

Okay, back to vectors. In this case, the *directions* can be also written as a vector $\vec{d}$, expressed as:

$$\vec{d} = 2\text{km}\,\hat{N} + 3\text{km}\,\hat{E}.$$

This is the mathematical expression that corresponds to the directions "walk 2 km north then 3 km east." Here, $\hat{N}$ is a *direction* and the number in front of the direction tells you the distance to walk in that direction.

Act 2: Equivalent directions

Later during your vacation, you decide to return to the location X. You arrive at the bus stop to find there is a slight problem. From your position, you can see a kilometre to the north, where a group of armed and threatening-looking men stand, waiting to ambush anyone who tries to cross what has now become a trail through the crops. Clearly the word has spread about X and constant visitors have drawn too much attention to the location.

Well, technically speaking, there is no problem at X. The problem lies on the route that starts north and travels through the ambush squad. Can you find an alternate route that leads to X?

```
"Use math, Luke! Use math!"
```

Recall the commutative property of addition for numbers: $a + b = b + a$. Maybe an analogous property holds for vectors? Indeed, this is the case:

$$\vec{d} = 2\text{km}\,\hat{N} + 3\text{km}\,\hat{E} = 3\text{km}\,\hat{E} + 2\text{km}\,\hat{N}.$$

The $\hat{N}$ directions and the $\hat{E}$ directions obey the commutative property. Since the directions can be followed in any order, you can first walk the 3 km east, then walk 2 km north and arrive at X again.

Act 3: Efficiency

It takes 5 km of walking to travel from the bus stop to X, and another 5 km to travel back to the bus stop. Thus, it takes a total of 10 km walking every time you want to go to X. Can you find a quicker route? What is the fastest way from the bus stop to the destination?

Instead of walking in the east and north directions, it would be quicker if you take the diagonal to the destination. Using Pythagoras' theorem you can calculate the length of the diagonal. When the side lengths are 3 and 2, the diagonal has length $\sqrt{3^2 + 2^2} = \sqrt{9+4} = \sqrt{13} = 3.60555\ldots$. The length of the diagonal route is just 3.6 km, Pay careful attention to the dot product and the cross product. Although they're called products, these operations behave much differently than taking the product of two numbers. Also note, there is no notion of vector division.

Vector algebra

Addition and subtraction Just like numbers, you can add vectors

$$\vec{v} + \vec{w} = (v_x, v_y) + (w_x, w_y) = (v_x + w_x, v_y + w_y),$$

subtract them

$$\vec{v} - \vec{w} = (v_x, v_y) - (w_x, w_y) = (v_x - w_x, v_y - w_y),$$

and solve all kinds of equations where the unknown variable is a vector. This is not a formidably complicated new development in mathematics. Performing arithmetic calculations on vectors simply requires **carrying out arithmetic operations on their components**. Given two vectors, $\vec{v} = (4, 2)$ and $\vec{w} = (3, 7)$, their difference is computed as $\vec{v} - \vec{w} = (4, 2) - (3, 7) = (1, -5)$.

Scaling We can also *scale* a vector by any number $\alpha \in \mathbb{R}$:

$$\alpha\vec{v} = (\alpha v_x, \alpha v_y),$$

where each component is multiplied by the scaling factor α. Scaling changes the length of a vector. If $\alpha > 1$ the vector will get longer, and if $0 \leq \alpha < 1$ then the vector will become shorter. If α is a negative number, the scaled vector will point in the opposite direction.

Length A vector's length is obtained from Pythagoras' theorem. Imagine a triangle with one side of length v_x and the other side of length v_y; the length of the vector is equal to the length of the triangle's hypotenuse:

$$\|\vec{v}\|^2 = v_x^2 + v_y^2 \qquad \Rightarrow \qquad \|\vec{v}\| = \sqrt{v_x^2 + v_y^2}\,.$$

A common technique is to scale a vector $\vec{v}$ by ~~the inverse of its length~~ one over its length $\frac{1}{\|\vec{v}\|}$ to obtain a unit-length vector that points in the same direction as $\vec{v}$:

$$\hat{v} \equiv \frac{\vec{v}}{\|\vec{v}\|} = \left(\frac{v_x}{\|\vec{v}\|}, \frac{v_y}{\|\vec{v}\|} \right).$$

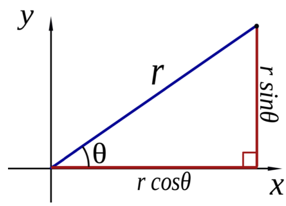
Unit vectors (denoted with a hat instead of an arrow) are useful when you want to describe only a direction in space without any specific length in mind. Verify that $\|\hat{v}\| = 1$.

Length and direction representation

So far, we've seen how to represent a vector in terms of its components. There is also another way of representing vectors: we can specify a vector in terms of its length $||\vec{v}||$ and its direction—the angle it ~~make~~ makes with the x-axis. For example, the vector $(1,1)$ can also be written as $\sqrt{2}\angle 45°$. This magnitude-and-direction notation is useful because it makes it easy to see the "size" of vectors. On the other hand, vector arithmetic operations are much easier to carry out in the component notation. We will use the following formulas for converting between the two notations.

To convert the length-and-direction vector $\|\vec{r}\|\angle\theta$ into an x-component and a y-component (r_x, r_y), use the formulas

$$r_x = \|\vec{r}\|\cos\theta \quad \text{and} \quad r_y = \|\vec{r}\|\sin\theta.$$



To convert from component notation (r_x, r_y) to length-and-direction $\|\vec{r}\|\angle\theta$, use

$$r = \|\vec{r}\| = \sqrt{r_x^2 + r_y^2} \qquad \text{and} \qquad \theta = \tan^{-1}\left(\frac{r_y}{r_x}\right).$$

Note that the second part of the equation involves the inverse tangent function. By convention, the function $\tan^{-1}$ returns values between $\pi/2$ $(90°)$ and $-\pi/2$ $(-90°)$. You must be careful when finding the θ of vectors with an angle outside of this range. Specifically, for vectors with $v_x < 0$, you must add π $(180°)$ to $\tan^{-1}(r_y/r_x)$ to obtain the correct θ.

Unit vector notation

As discussed above, we can think of a vector $\vec{v} = (v_x, v_y, v_z)$ as a command to "go a distance v_x in the x-direction, a distance v_y in the y-direction, and v_z in the z-direction."

To write this set of commands more explicitly, we can use multiples of the vectors $\hat{\imath}, \hat{\jmath}$, and $\hat{k}$. These are the unit vectors pointing in the x, y, and z directions, respectively:

$$\hat{\imath} = (1,0,0), \qquad \hat{\jmath} = (0,1,0), \quad \text{and} \quad \hat{k} = (0,0,1).$$

Any number multiplied by $\hat{\imath}$ corresponds to a vector with that number in the first coordinate. For example, $3\hat{\imath} \equiv (3,0,0)$. Similarly, $4\hat{\jmath} \equiv (0,4,0)$ and $5\hat{k} \equiv (0,0,5)$.

In physics, we tend to perform a lot of numerical calculations with vectors; to make things easier, we often use unit vector notation:

$$v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{k} \qquad \Leftrightarrow \qquad \vec{v} \qquad \Leftrightarrow \qquad (v_x, v_y, v_z).$$

The addition rule remains the same for the new notation:

$$\underbrace{2\hat{\imath} + 3\hat{\jmath}}_{\vec{v}} \;+\; \underbrace{5\hat{\imath} - 2\hat{\jmath}}_{\vec{w}} \;=\; \underbrace{7\hat{\imath} + 1\hat{\jmath}}_{\vec{v}+\vec{w}}.$$

It's the same story repeating all over again: we need to add $\hat{\imath}$s with $\hat{\imath}$s, and $\hat{\jmath}$s with $\hat{\jmath}$s.

Examples

Simple example

Compute the sum $\vec{s} = 4\hat{\imath} + 5\angle 30^\circ$. Express your answer in the length-and-direction notation.

Since we want to carry out an addition, and since addition is performed in terms of the components, our first step is to convert $5\angle 30^\circ$ into component notation. We find $\cancel{5\angle 30^\circ = (5\cos 30^\circ)\hat{\imath} + (5\sin 30^\circ)\hat{\jmath} = 5\frac{\sqrt{3}}{2}\hat{\imath} + \frac{5}{2}}$ We can now compute the sum:

$$\vec{s} \;=\; 4\hat{\imath} \;+\; 5\tfrac{\sqrt{3}}{2}\hat{\imath} + \tfrac{5}{2}\hat{\jmath} \;=\; (4 + 5\tfrac{\sqrt{3}}{2})\hat{\imath} + (\tfrac{5}{2})\hat{\jmath}.$$

The x-component of the sum is $s_x = (4 + 5\frac{\sqrt{3}}{2})$, and the y-component of the sum is $s_y = (\frac{5}{2})$. To express the answer as a length and a direction, we compute the length $\|\vec{s}\| = \sqrt{s_x^2 + s_y^2} = 8.697$ and the direction $\tan^{-1}(s_y/s_x) = 16.7^\circ$. The answer is $\vec{s} = 8.697\angle 16.7^\circ$.

Vector addition example

You're heading to physics class after a "safety meeting" with a friend, and are looking forward to two hours of finding absolute amazement and awe in the laws of Mother Nature. As it turns out, there is no enlightenment to be had that day because there is going to be an in-class midterm. The first question involves a block sliding down an incline. You look at it, draw a little diagram, and then wonder how the hell you are going to find the net force acting on the block. The three forces acting on the block are ~~$\vec{W} = 30\angle -90^\circ$, $\vec{N} = 200\angle -290^\circ$, and $\vec{F}_f = 50\angle 60^\circ$~~$\vec{W} = 300\angle -90^\circ$, $\vec{N} = 260\angle 120^\circ$, and $\vec{F}_f = 50\angle 30^\circ$.

You happen to remember the net force formula:

$$\sum \vec{F} = \vec{F}_{\text{net}} = m\vec{a} \qquad [\text{ Newton's } 2^{\text{nd}} \text{ law }].$$

You get the feeling Newton's 2nd law is the answer to all your troubles. You sense this formula is certainly the key because you saw the keyword "net force" when reading the question, and notice "net force" also appears in this very equation.

3.4 Vector products

If addition of two vectors $\vec{v}$ and $\vec{w}$ corresponds to the addition of their components $(v_x+w_x, v_y+w_y, v_z+w_z)$, you might logically think that the product of two vectors will correspond to the product of their components (v_xw_x, v_yw_y, v_zw_z), however, this way of multiplying vectors is not used in practice. Instead, we use the dot product and the cross product.

The *dot product* tells you how similar two vectors are to each other:

$$\vec{v}\cdot\vec{w} \equiv v_xw_x + v_yw_y + v_zw_z \equiv \|\vec{v}\|\|\vec{w}\|\cos(\varphi) \quad \in \mathbb{R},$$

where φ is the angle between the two vectors. The factor $\cos(\varphi)$ is largest when the two vectors point in the same direction because the angle between them will be $\varphi = 0$ and $\cos(0) = 1$.

The exact formula for the *cross product* is more complicated so I will not show it to you just yet. What is important to know is that the cross product of two vectors is another vector:

$$\vec{v}\times\vec{w} = \{\text{ a vector perpendicular to both } \vec{v} \text{ and } \vec{w}\ \} \quad \in \mathbb{R}^3.$$

If you take the cross product of one vector pointing in the x-direction with another vector pointing in the y-direction, the result will be a vector in the z-direction.

Dot product

The *dot product* takes two vectors as inputs and produces a real number as output:

$$\cdot : \mathbb{R}^3 \times \mathbb{R}^3 \quad \to \quad \mathbb{R}.$$

The dot product between two vectors can be computed using either the algebraic formula

$$\vec{v}\cdot\vec{w} \equiv v_xw_x + v_yw_y + v_zw_z,$$

or the geometrical formula

$$\vec{v}\cdot\vec{w} \equiv \|\vec{v}\|\|\vec{w}\|\cos(\varphi),$$

where φ is the angle between the two vectors. ~~This operation~~

The dot product is also known as the *inner* product or *scalar* product. The name *scalar* comes from the fact that the result of the dot product is a scalar number—a number that does not change when the basis changes.

~~The signature~~ We can combine the algebraic and the geometric formulas for the dot product ~~operation is The dot product takes two~~

~~vectors as inputs and produces a real number as output~~to obtain the formula

$$\cos(\varphi) = \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|} = \frac{v_x w_x + v_y w_y + v_z w_z}{\|\vec{v}\|\|\vec{w}\|} \quad \text{and} \quad \varphi = \cos^{-1}(\cos(\varphi)).$$

Thus, it's possible to find the angle between two vectors if we know their components.

The geometric factor $\cos(\varphi)$ depends on the relative orientation of the two vectors as follows:

- If the vectors point in the same direction, then $\cos(\varphi) = \cos(0°) = 1$ and so $\vec{v}\cdot\vec{w} = \|\vec{v}\|\|\vec{w}\|$.
- If the vectors are perpendicular to each other, then $\cos(\varphi) = \cos(90°) = 0$ and so $\vec{v}\cdot\vec{w} = \|\vec{v}\|\|\vec{w}\|(0) = 0$.
- If the vectors point in exactly opposite directions, then $\cos(\varphi) = \cos(180°) = -1$ and so $\vec{v}\cdot\vec{w} = -\|\vec{v}\|\|\vec{w}\|$.

Cross product

The *cross product* takes two vectors as inputs and produces another vector as the output:

$$\times : \mathbb{R}^3 \times \mathbb{R}^3 \quad\to\quad \mathbb{R}^3.$$

Because the output of this operation is a vector, we sometimes refer to the cross product as the *vector* product.

The cross products of individual basis elements are defined as follows:

$$\hat{\imath}\times\hat{\jmath} = \hat{k}, \quad \hat{\jmath}\times\hat{k} = \hat{\imath}, \quad \hat{k}\times\hat{\imath} = \hat{\jmath}.$$

The cross product is *anti-symmetric* in its inputs, which means swapping the order of the inputs introduces a negative sign in the output:

$$\hat{\jmath}\times\hat{\imath} = -\hat{k}, \quad \hat{k}\times\hat{\jmath} = -\hat{\imath}, \quad \hat{\imath}\times\hat{k} = -\hat{\jmath}.$$

I bet you had never seen a product like this before. Most likely, the products you've seen in math have been *commutative*, which means the order of the inputs doesn't matter. The product of two numbers is commutative $ab = ba$, and the dot product is commutative $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$, but the cross product of two vectors is *non-commutative* $\hat{\imath}\times\hat{\jmath} \neq \hat{\jmath}\times\hat{\imath}$.

For two arbitrary vectors $\vec{a} = (a_x, a_y, a_z)$ and $\vec{b} = (b_x, b_y, b_z)$, the cross product is calculated as

$$\vec{a}\times\vec{b} = (a_y b_z - a_z b_y,\ a_z b_x - a_x b_z,\ a_x b_y - a_y b_x)\,.$$

The cross product's output has a length that is proportional to the sin of the angle between the vectors:

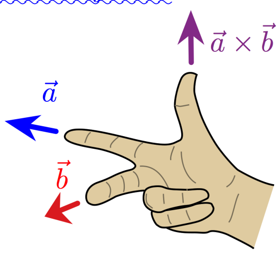
$$\|\vec{a} \times \vec{b}\| = \|\vec{a}\|\|\vec{b}\| \sin(\varphi).$$

The direction of the vector $(\vec{a} \times \vec{b})$ is perpendicular to both $\vec{a}$ and $\vec{b}$.

The right-hand rule

Consider the plane formed by the vectors $\vec{a}$ and $\vec{b}$. There are actually *two* vectors that are perpendicular to this plane: one above the plane and one below the plane. We use the *right-hand rule* to figure out which of these vectors corresponds to the cross product $\vec{a} \times \vec{b}$.

When your index finger points in the same direction as the vector $\vec{a}$ and your middle finger points in the direction of $\vec{b}$, your thumb will point in the direction of $\vec{a} \times \vec{b}$. The relationship encoded in the right-hand rule matches the relationship between the standard basis vectors: $\hat{\imath} \times \hat{\jmath} = \hat{k}$.



Links

[A nice illustration of the cross product]
http://1ucasvb.tumblr.com/post/76812811092/

3.5 Complex numbers

By now, you've heard about complex numbers $\mathbb{C}$. The word "complex" is an intimidating word. Surely it must be a complex task to learn about the complex numbers. That may be true in general, but it helps if you know about vectors. Complex numbers are similar to two-dimensional vectors $\vec{v} \in \mathbb{R}^2$. We add and subtract complex numbers like vectors. Complex numbers also have components, length, and "direction." If you understand vectors, you will understand complex numbers at almost no additional mental cost.

We'll begin with a practical problem.

Example

Suppose you~~are~~'re asked to solve the following quadratic equation:

$$x^2 + 1 = 0.$$

You're looking for a number x, such that $x^2 = -1$. If you are only allowed to give real answers (the set of real numbers is denoted $\mathbb{R}$), then there is no answer to this question. In other words, this equation has no solutions. Graphically speaking, this is because the quadratic function $f(x) = x^2 + 1$ does not cross the x-axis.

However, we're not ~~going to take nothing as an answer. We will~~ taking no for an answer! If we insist on solving for x in the equation $x^2 + 1 = 0$, we can imagine a new number ~~called~~ i that satisfies $i^2 = -1$. We call i the unit imaginary number. The solutions to the equation are therefore $x_1 = i$ and $x_2 = -i$. There are two solutions because the equation was quadratic. We can check that $i^2 + 1 = -1 + 1 = 0$ and also $(-i)^2 + 1 = (-1)^2 i^2 + 1 = i^2 + 1 = 0$.

Thus, while the equation $x^2 + 1 = 0$ has no real solutions, it *does* have solutions if we allow the answers to be ~~complex~~ imaginary numbers.

Definitions

Complex numbers have a real part and an imaginary part:

- i: the unit imaginary number $i \equiv \sqrt{-1}$ or $i^2 = -1$
- bi: an imaginary number that is equal to b times i
- $\mathbb{R}$: the set of real numbers
- $\mathbb{C}$: the set of complex numbers $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$
- $z = a + bi$: a complex number
- $\mathrm{Re}\{z\} = a$: the real part of z
- $\mathrm{Im}\{z\} = b$: the imaginary part of z
- $\bar{z}$: the *complex conjugate* of z. If $z = a + bi$, then $\bar{z} = a - bi$.

The polar representation of complex numbers:

- ~~$z = |z|\angle\phi_z = |z|\cos\phi_z + i|z|\sin\phi_z$~~ $z = |z|\angle\varphi_z = |z|\cos\varphi_z + i|z|\sin\varphi_z$
- $|z| = \sqrt{\bar{z}z} = \sqrt{a^2 + b^2}$: the *magnitude* of $z = a + bi$
- ~~$\phi_z = \tan^{-1}(b/a)$: the phase~~ $\varphi_z = \tan^{-1}(b/a)$: the *phase* or *argument* of $z = a + bi$
- ~~$\mathrm{Re}\{z\} = |z|\cos\phi_z$~~ $\mathrm{Re}\{z\} = |z|\cos\varphi_z$
- ~~$\mathrm{Im}\{z\} = |z|\sin\phi_z$~~ $\mathrm{Im}\{z\} = |z|\sin\varphi_z$

Formulas

Addition and subtraction

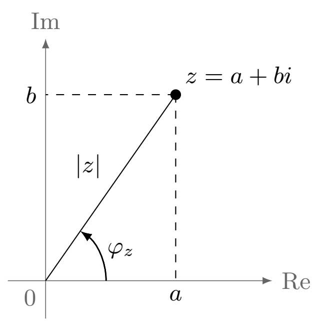
Just as we performed the addition of vectors component by component, we perform addition on complex numbers by adding the real parts together and adding the imaginary ~~part~~ parts together:

$$(a + bi) + (c + di) = (a + c) + (b + d)i.$$

Polar representation

We can give a geometrical interpretation of the complex numbers by extending the real number line into a two-dimensional plane called the *complex plane*. The horizontal axis in the complex plane measures the *real* part of the number. The vertical axis measures the *imaginary* part. Complex numbers are vectors in the complex plane.

It is possible to represent any complex number $z = a + bi$ in terms of its *magnitude* and its *phase*:



$$z = |z|\angle\varphi_z = \underbrace{|z|\cos\varphi_z}_{a} + \underbrace{|z|\sin\varphi_z}_{b}\, i.$$

The magnitude of a complex number $z = a + bi$ is

$$|z| = \sqrt{a^2 + b^2}.$$

This corresponds to the *length* of the vector ~~which~~ that represents the complex number in the complex plane. The formula is obtained by using Pythagoras' theorem.

The *phase*, also known as the *argument* of the complex number ~~is:~~ $z = a + bi$ is

$$\varphi_z \equiv \arg z \;=\; \texttt{atan2(}b\texttt{,}a\texttt{)} \;\overset{\dagger}{=}\; \tan^{-1}(b/a).$$

The phase corresponds to the angle z forms with the real axis. Note the equality labelled † is true only when $a > 0$, because the function $\tan^{-1}$ always returns numbers in the range $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Manual corrections of the output of $\tan^{-1}(b/a)$ are required for complex numbers with $a < 0$.

Some programming languages provide the 2-input math function `atan2(y,x)` that correctly computes the angle the vector (x, y) makes with the x-axis in all four quadrants. Complex numbers behave like 2-dimensional vectors so you can use `atan2` to compute their phase.

Complex numbers have vector-like properties like magnitude and phase, but we can also do other operations with them that are not defined for vectors. The set of complex numbers $\mathbb{C}$ is a *field*. This means, in addition to the addition and subtraction operations, we can also perform multiplication and division with complex numbers.

Multiplication

The product of two complex numbers is computed using the usual rules of algebra:

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$

In the polar representation, the product formula is

$$(p\angle\phi)(q\angle\psi) = pq\angle(\phi + \psi).$$

To multiply two complex numbers, multiply their magnitudes and add their phases.

~~Cardano's example~~Division

Let's look at the procedure for dividing complex numbers:

$$\frac{(a+bi)}{(c+di)} = \frac{(a+bi)}{(c+di)}\frac{(c-di)}{(c-di)} = (a+bi)\frac{(c-di)}{(c^2+d^2)} = (a+bi)\frac{\overline{c+di}}{|c+di|^2}.$$

In other words, to divide the number z by the complex number s, compute $\bar{s}$ and $|s|^2 = s\bar{s}$ and then use

$$z/s = z\frac{\bar{s}}{|s|^2}.$$

You can think of $\frac{\bar{s}}{|s|^2}$ as being equivalent to s^{-1}.

Cardano's example One of the earliest examples of reasoning involving complex numbers was given by Gerolamo Cardano in his 1545 book *Ars Magna*. Cardano wrote, "If someone says to you, divide 10 into two parts, one of which multiplied into the other shall produce 40, it is evident that this case or question is impossible." We want to find numbers x_1 and x_2 such that $x_1 + x_2 = 10$ and $x_1x_2 = 40$. This sounds kind of impossible. Or is it?

"Nevertheless," Cardano said, "we shall solve it in this fashion:

$$x_1 = 5 + \sqrt{15}i \quad \text{and} \quad x_2 = 5 - \sqrt{15}i."$$

When you add $x_1 + x_2$ you obtain 10. When you multiply the two numbers the answer is

$$\begin{aligned} x_1 x_2 &= \left(5+\sqrt{15}i\right)\left(5-\sqrt{15}i\right) \\ &= 25 - 5\sqrt{15}i + 5\sqrt{15}i - \sqrt{15}^2 i^2 = 25 + 15 = 40. \end{aligned}$$

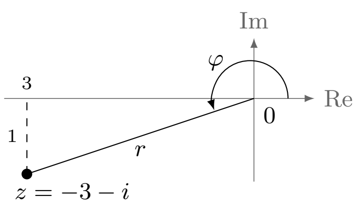
Hence $5+\sqrt{15}i$ and $5-\sqrt{15}i$ are two numbers whose sum is 10 and whose product is 40.

~~**Example**~~

Compute the product of i and -1. Both i and -1 have a magnitude of 1 ~~,~~ but different phases. The phase of i ~~has phase~~ is $\frac{\pi}{2}$ (90°), while -1 has phase π (180°). ~~Consider the~~ The product of these two numbers ~~:~~ is

$$(i)(-1) = (1\angle\tfrac{\pi}{2})(1\angle\pi) = 1\angle(\tfrac{\pi}{2}+\pi) = 1\angle\tfrac{3\pi}{2} = -i.$$

Multiplication by i is effectively a rotation by ~~90 degrees leftward~~ $\frac{\pi}{2}$ (90°) to the left.



Find the polar representation of $z = -3 - i$ and compute z^6.

Let's ~~look at the procedure for dividing complex numbers: In other words, to divide the number~~ denote the polar representation of z by ~~the complex number s, compute $\bar{s}$~~ $z = r\angle\varphi$. We find $r = \sqrt{3^2+1^2} = \sqrt{10}$ and ~~$|s|^2 = s\bar{s}$ and then use You can think of $\frac{\bar{s}}{|s|^2}$ as being equivalent to s^{-1}~~ $\varphi = \tan^{-1}(\frac{1}{3}) + \pi = 0.322 + \pi$.

Using the polar representation, we can easily compute z^6:

$$z^6 = r^6\angle(6\varphi) = (\sqrt{10})^6\angle 6(0.322+\pi) = 10^3\angle 1.932 + 6\pi = 10^3\angle 1.932.$$

Note we can ignore multiples of 2π in the phase. In component form, z^6 is equal to $1000\cos(1.932) + 1000\sin(1.932)i = -353.4 + 935.5i$.

Fundamental theorem of algebra

The solutions to *any* polynomial equation ~~$a_0 + a_1 x + \cdots a_n x^n = 0$~~ $a_0 + a_1 x + \cdots + a_n x^n = 0$ are of the form

$$z = a + bi.$$

In other words, any polynomial $P(x)$ of n^{th} degree can be written as

$$P(x) = (x - z_1)(x - z_2)\cdots(x - z_n),$$

where $z_i \in \mathbb{C}$ are the polynomial's *complex* roots. Before today, you might have said the equation $x^2 + 1 = 0$ has no solutions. Now you know its solutions are the complex numbers $z_1 = i$ and $z_2 = -i$.

The theorem is "fundamental" because it tells us we won't ever need to invent any "fancier" set of numbers to solve polynomial equations. Recall that each set of numbers is associated with a different class of equations. The natural numbers $\mathbb{N}$ appear as solutions of the equation $m + n = x$, where m and n are natural numbers (denoted $m, n \in \mathbb{N}$). The integers $\mathbb{Z}$ are the solutions to equations of the form $x + m = n$, where $m, n \in \mathbb{N}$. The rational numbers $\mathbb{Q}$ are necessary to solve for x in $mx = n$, with $m, n \in \mathbb{Z}$. To find the solutions of $x^2 = 2$, we need the real numbers $\mathbb{R}$. The process of requiring new types of numbers for solving more complicated types of equations stops at $\mathbb{C}$; any polynomial equation—no matter how complicated it is—has solutions that are complex numbers $\mathbb{C}$.

Euler's formula

You already know $\cos\theta$ is a shifted version of $\sin\theta$, so it's clear these two functions are related. It turns out the exponential function is also related to sin and cos. Lo and behold, we have Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta\,.$$

Inputting an imaginary number to the exponential function outputs a complex number that contains both cos and sin. Euler's formula gives us an alternate notation for the polar representation of complex numbers: $\cancel{z = |z|\angle\phi_z = |z|e^{i\phi_z}}$ $z = |z|\angle\varphi_z = |z|e^{i\varphi_z}$.

If you want to impress your friends with your math knowledge, plug $\theta = \pi$ into the above equation to find

$$e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1,$$

which can be rearranged into the form, $e^{\pi i} + 1 = 0$. This equation shows a relationship between the five most important numbers in all of mathematics: Euler's number $e = 2.71828\ldots$, $\pi = 3.14159\ldots$, the imaginary number i, 1, and zero. It's kind of cool to see all these important numbers reunited in one equation, don't you agree?

De Moivre's theorem

By replacing θ in Euler's formula with $n\theta$, we obtain de Moivre's theorem:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

De Moivre's Theorem makes sense if you think of the complex number $z = e^{i\theta} = \cos\theta + i\sin\theta$, raised to the n^{th} power:

$$(\cos\theta + i\sin\theta)^n = z^n = (e^{i\theta})^n = e^{in\theta} = \cos n\theta + i\sin n\theta.$$

Setting $n = 2$ in de Moivre's formula, we can derive the double angle formulas (page 73) as the real and imaginary parts of the following equation:

$$(\cos^2\theta - \sin^2\theta) + (2\sin\theta\cos\theta)i = \cos(2\theta) + \sin(2\theta)i.$$

Links

[Mini tutorial on the complex numbers]
http://paste.lisp.org/display/133628

Concepts

The basic concepts of kinematics in two dimensions are:

- $\hat{x}, \hat{y}$: the xy-coordinate system
- t: time, measured in seconds
- $\vec{r}(t) \equiv (x(t), y(t))$: the position vector of the object at time t
- $\vec{v}(t) \equiv (v_x(t), v_y(t))$: the velocity vector of the object
- $\vec{a}(t) \equiv (a_x(t), a_y(t))$: the acceleration vector of the object

We will use the following terminology when analyzing the motion of an object that starts from an *initial* ~~point~~ position and travels to a *final* position:

- $t_i = 0$: the initial time
- t_f: the final time
- $\vec{v}_i = (v_x(0), v_y(0)) = (v_{ix}, v_{iy})$: the initial velocity at $t = 0$
- $\vec{r}_i = (x(0), y(0)) = (x_i, y_i)$: the initial position at $t = 0$
- $\vec{r}_f = \vec{r}(t_f) = (x(t_f), y(t_f)) = (x_f, y_f)$: the position at $t = t_f$

Definitions

Motion in two dimensions

We use the position vector $\vec{r}(t)$ to describe the x and y coordinates of the projectile as a function of time:

$$\vec{r}(t) \equiv (x(t), y(t)).$$

We use x to describe the horizontal distance travelled by the projectile and y to describe the height of the projectile.

The velocity of the projectile is the derivative of its position:

$$\vec{v}(t) = \frac{d}{dt}(\vec{r}(t)) = \left(\frac{dx(t)}{dt}, \frac{dy(t)}{dt}\right) = (v_x(t), v_y(t)).$$

The initial velocity is an important parameter of the motion:

$$\vec{v}(0) = (v_x(0), v_y(0)) = (v_{ix}, v_{iy}) = (\|\vec{v}_i\|\cos\theta, \|\vec{v}_i\|\sin\theta) = \|\vec{v}_i\|\angle\theta.$$

The initial velocity vector can be expressed as components (v_{ix}, v_{iy}), or in the length-and-direction form $\|\vec{v}_i\|\angle\theta$, where θ measures the angle between $\vec{v}_i$ and the x-axis.

~~The~~ On Earth, the acceleration of the projectile is

$$\vec{a}(t) = \frac{d}{dt}(\vec{v}(t)) = (a_x(t), a_y(t)) = (0, -9.81).$$

You flick r with ~~you~~ your finger at an initial velocity of $\vec{v}_i = (10.5, 0)$[m/s] and ~~the particle~~ it flies straight into the garbage bin. Success!

Freedom and democracy

An American F-18 is flying above Iraq. It is carrying two bombs. One bomb is named "freedom" and weighs 200[kg]; the other is called "democracy" and ~~packs a mass of~~ weighs 500[kg]. If the plane is flying horizontally with speed $v_i = 300$[m/s] and drops both bombs from a height of 2000[m], how far will ~~the bombs travel before they hit~~ each bomb travel horizontally before it hits the ground? Which city will get freedom and which city will get democracy?

The equations of motion for the bombs are

$$x(t) = v_{ix}t + x_i = 300t + 0$$

and

$$y(t) = \tfrac{1}{2}(-9.81)t^2 + v_{iy}t + y_i = -4.9t^2 + 2000.$$

To find where the bombs will land, the first step is to calculate the time of flight. We solve for t_f in the equation $y(t_f) = 0$ and find $t_f = 20.20$[s]. We can then find the final x-position where the bombs hit the ground from the first equation: $x_f = x(20.20) = 6060$[m]. Both bombs hit the ~~same town, the one~~ point located 6.06[km] from the launch point. Observe that the bombs' masses did not play any part in the ~~final~~ equations of motion.

Let's be real. The scenario at hand is essentially what the people in Washington are talking about when they say they are bringing freedom and democracy to the Middle East. A monstrous amalgamation of warmongering corporations, weak politicians, and special-interest lobby groups make a complete mockery of the political process. In order to see an end to world conflict, I think the entire military-industrial complex needs to be dismantled. How can we stop them, you ask? In my opinion, the best way to fight the System is not to work for the System. If some recruiters from that sector comes to offer you a job one day because you're a math expert, tell them to scram.

Interception

With all those people launching explosive projectiles at each other, a need develops for *interception* systems that can throw counter-projectiles at the incoming projectiles and knock them out of the air.

Normal force

The *normal* force is the force between two surfaces in contact. In this context, the word *normal* means "perpendicular to the surface of." The reason my coffee mug is not falling to the floor right now is that the table exerts a normal force $\vec{N}$ on the mug, keeping it in place.

Force of friction

In addition to the normal force between surfaces, there is also the force of friction $\vec{F}_f$, which acts to impede any sliding motion between the surfaces. There are two kinds of friction forces, and both are proportional to the amount of normal force between the surfaces:

$$\max\{\vec{F}_{fs}\} = \mu_s\|\vec{N}\| \ \ \text{(static)}, \quad \text{and} \quad \vec{F}_{fk} = \mu_k\|\vec{N}\| \ \ \text{(kinetic)},$$

where μ_s and μ_k are the static and ~~dynamic~~ kinetic *friction coefficients*. It makes sense that the force of friction should be proportional to the magnitude of the normal force $\|\vec{N}\|$, since the harder the two surfaces push against each other, the more difficult it becomes to make them slide. The above equations give mathematical precision to this intuitive logic.

The static force of friction acts on objects that are not moving. It describes the *maximum* amount of friction that can exist between two objects. If a horizontal force greater than $F_{fs} = \mu_s N$ is applied to the object, then it will start to slip. The kinetic force of friction acts when two objects are sliding relative to each other. It always acts in the direction opposite to the motion.

Tension

A force can also be exerted on an object remotely by attaching a rope to the object, and pulling the rope. The force exerted on the object will be equal to the rope's *tension* $\vec{T}$. Note that tension always pulls *away* from an object: you can pull but you can't push a dog by its leash.

Discussion

Viewing the interactions between objects in terms of the forces that act between them gives us a powerful tool for thinking and analyzing physics problems. The following section shows you how to draw force diagrams that account for all the forces acting on an object.

Formulas

Newton's 2nd law

The net force $\vec{F}_{\text{net}}$ is the sum of the forces acting on an object~~,~~. Assuming the object is rigid, the location where the forces act on the object is not important, so we can assume all forces act at the object's centre of mass.

The net force acting on an object, divided by the object's mass, gives the acceleration of the object:

$$\sum \vec{F} \equiv \vec{F}_{\text{net}} = m\vec{a}. \tag{4.7}$$

Vector components

If a vector $\vec{v}$ makes an angle θ with the x-axis, then

$$v_x = \|\vec{v}\| \cos\theta \qquad \text{and} \qquad v_y = \|\vec{v}\| \sin\theta.$$

The vector $v_x\hat{\imath}$ corresponds to the part of $\vec{v}$ that points in the x-direction.

Shortly, I'll be asking you over and over again to

find the component of $\vec{F}$ in the ? direction,

which is another way of asking you to find the number $v_?$.

The answer is usually equal to the length $\|\vec{F}\|$ multiplied by either cos or sin or sometimes -1, **depending on way the coordinate system is chosen**. So don't guess. Look at the coordinate system. If the vector points in the direction where x increases, then v_x should be a positive number. If $\vec{v}$ points in the opposite direction, then v_x should be negative.

To add forces $\vec{F}_1$ and $\vec{F}_2$, you need to add their components:

$$\vec{F}_1 + \vec{F}_2 = (F_{1x}, F_{1y}) + (F_{1x}, F_{2y}) = (F_{1x} + F_{2x}, F_{1y} + F_{2y}) = \vec{F}_{\text{net}}.$$

However, instead of dealing with vectors in the bracket notation, when solving force diagrams it is easier to simply write the x equation on one line, and the y equation on a separate line below it:

$$F_{\text{net},x} = F_{1x} + F_{2x},$$

$$F_{\text{net},y} = F_{1y} + F_{2y}.$$

It's a good idea to always write those two equations together as a block, so it's clear the first row represents the x dimension and the second row represents the y dimension for the same problem.

Other types of problems

Each of the previous examples asked you to find the acceleration, but sometimes a problem might give you the acceleration and ask you to solve for a different unknown. Regardless of what you must solve for, you should always start with a diagram and a sum-of-the-forces template. Once these equations are in front of you, you'll be able to reason through the problem more easily.

Experiment

You remove the spring from a retractable pen, and from the spring you suspend an object of known mass—say a 100[g] chocolate bar. With a ruler, you measure how much the spring stretches in the process. What is the spring constant k?

Discussion

In previous sections we discussed the *kinematics* problem of finding an object's position $x(t)$ given its acceleration function $a(t)$, and given the initial conditions x_i and v_i. In this section we studied the *dynamics* problem, which involves drawing force diagrams and calculating the net force acting on an object. Understanding these topics means you fully understand Newton's equation $F = ma$, which is perhaps the most important equation in this book.

We can summarize the entire procedure for predicting the position of an object $x(t)$ from first principles in the following equation:

$$\frac{1}{m}\underbrace{\left(\sum \vec{F} = \vec{F}_{\text{net}}\right)}_{\text{dynamics}} = \underbrace{\vec{a}(t) \quad \xrightarrow{\vec{v}_i + \int dt} \quad \vec{v}(t) \quad \xrightarrow{\vec{r}_i + \int dt} \quad \vec{r}(t)}_{\text{kinematics}}.$$

The left-hand side calculates the net force acting on an object, which is the *cause* of acceleration. The right-hand side indicates how we can calculate the position vector $\vec{r}(t)$ starting from the acceleration and the initial conditions. If you know the forces acting on any object (rocks, projectiles, cars, stars, planets, etc.) then you can predict the object's motion using this equation, which is pretty cool.

So far we discussed one approach for analyzing the motion of objects. Calculating the forces and the acceleration of objects, then using integration to find the position function $\vec{r}(t)$ is a very useful approach for solving physics problems. There are several other ways of looking at the motion of objects that are equally useful and provide us with different insights. In the next two sections, we'll discuss how to model physical situations in terms of momentum and energy.

4.5 Momentum

A collision between two objects creates a sudden spike in the contact force between them, which can be difficult to measure and quantify. It is not possible to use Newton's law $F = ma$ to predict the accelerations that occur during collisions. To predict the motion of the objects after the collision, we need a *momentum* calculation. According to the law of conservation of momentum, the total amount of momentum before and after the collision is the same. Once we know the momenta of the objects before the collision, it becomes possible to calculate their momenta after the collision, and from this determine their subsequent motion.

To illustrate the importance of momentum, consider the following situation. Say you have a 1[g] ~~piece of paper~~ paper ball and a 1000[kg] car moving at the same speed of 100[km/h]. Which of the two objects would you rather be hit by? Momentum, denoted $\vec{p}$, is the precise physical concept that measures the *quantity* of motion. An object of mass m moving with velocity $\vec{v}$ has a momentum of $\vec{p} \equiv m\vec{v}$. Momentum plays a key role in collisions. Your gut feeling about the piece of paper and the car is correct. The car weighs $1000 \times 1000 = 10^6$ times more than the piece of paper, so the car has 10^6 times more momentum when moving at the same speed. Colliding with the car will "hurt" one-million times more than colliding with the piece of paper, even though both objects approach ~~have~~ at the same velocity.

In this section, we'll learn how to use the law of conservation of momentum to predict the outcomes of collisions.

Concepts

- m: the *mass* of the moving object
- $\vec{v}$: the *velocity* of the moving object
- $\vec{p} = m\vec{v}$: the *momentum* of the moving object
- $\sum \vec{p}_{\text{in}}$: the sum of the momenta of particles before a collision
- $\sum \vec{p}_{\text{out}}$: the sum of the momenta of particles after a collision

Definition

The momentum of a moving object is equal to the velocity of the object multiplied by its mass:

$$\vec{p} = m\vec{v} \qquad [\text{kg m/s}]. \tag{4.8}$$

If an object's velocity is $\vec{v} = 20\hat{\imath} = (20, 0)$[m/s] and its mass is 100[kg], then its momentum is $\vec{p} = 2000\hat{\imath} = (2000, 0)$[kg m/s].

learn the rules for converting one energy into another. The key idea to keep in mind is the principle of *total energy conservation*, which says that in any physical process, the sum of the initial energies is equal to the sum of the final energies.

Example

You drop a ball from a height h[m] and want to predict its speed just before it hits the ground. Through the kinematics approach, you would set up the general equation of motion,

$$v_f^2 = v_i^2 + 2a(y_f - y_i),$$

substitute $y_i = h$, $y_f = 0$, $v_i = 0$, and $a = -g$, and solve for the ball's final ~~velocity~~ speed at impact v_f. The answer is $v_f = \sqrt{2gh}$[m/s].

Alternately, we can use an energy calculation. The ball starts from a height h, which means it has $U_i = mgh$[J] of potential energy. As the ball falls, potential energy is converted into kinetic energy. Just before the ball hits the ground, its final kinetic energy is equal to the initial potential energy: $K_f = U_i$. Since the formula for kinetic energy is $K = \frac{1}{2}mv^2$[J], we have $\frac{1}{2}mv_f^2 = mgh$. We cancel the mass on both sides of the equation and solve for v_f to obtain $v_f = \sqrt{2gh}$[m/s].

Both methods of solving the example problem lead us to the same conclusion, but the energy reasoning is arguably more intuitive than blindly plugging values into a formula. In science, it is really important to know different ways of arriving at the same answer. Knowing about these alternate routes will allow you to check your answers and better understand concepts.

Concepts

Energy is measured in Joules [J] and it arises in several contexts:

- K = **kinetic energy**: the type of energy objects have by virtue of their motion
- W = **work**: the amount of energy an external force adds or subtracts from a system. Positive work corresponds to energy added to the system while negative work corresponds to energy withdrawn from the system.
- U_g = **gravitational potential energy**: the energy an object has by virtue of its position above the ground. We say this energy is *potential* because it is a form of *stored work*. Potential energy corresponds to the amount of work the force of gravity will add to an object when the object falls to the ground.

Kinetic energy

A moving object has energy $K = \frac{1}{2}m\|\vec{v}\|^2$[J], called *kinetic* energy from the Greek word for motion, *kinema*.

Note that velocity $\vec{v}$ and speed $\|\vec{v}\|$ are not the same as energy. Suppose you have two objects of the same mass, and one is moving two times faster than the other. The faster object will have twice the velocity of the slower object, and four times more kinetic energy.

Work

When hiring movers to help you move, you must pay them for the *work* they do. Work is the product of the amount of force needed for the move and the distance of the move. When the move requires a lot of force, more work will be done. And the bigger the displacement (think moving to the South Shore versus moving next door), the more money the movers will ask for.

The amount of work done by a force $\vec{F}$ on an object that moves along some path p is given by

$$W = \int_p \vec{F}(x) \cdot d\vec{x}.$$

The integral ~~account~~ accounts for the fact that the force's magnitude and direction might change along the path of motion.

If the force is constant and the displacement path is a straight line, the formula for work simplifies to

$$W = \int_0^d \vec{F} \cdot d\vec{x} = \vec{F} \cdot \int_0^d d\vec{x} = \vec{F} \cdot \vec{d} = \|\vec{F}\|\|\vec{d}\| \cos\theta. \qquad (4.11)$$

Note the use of the dot product to obtain only the part of $\vec{F}$ that is pushing in the direction of the displacement $\vec{d}$. A force that acts perpendicular to the displacement produces no work, since it neither speeds nor slows the object's motion.

Potential energy is stored work

Some kinds of work are just a waste of time, like working at a job you despise. You work and you get your paycheque, but you don't learn anything useful at the end of the day. Other kinds of work leave you with some useful resource at the end of the work day—they grow your human potential.

In physics, we make a similar distinction. Some types of work, like work against friction, are called *dissipative* since they simply waste energy. Other kinds of work are called *conservative* since the work performed isn't lost but converted into *potential energy*.

higher at (A) than it is at (B), so we write

$$0 + mg|y_i|\sin\theta = \frac{1}{2}mv_B^2 + 0.$$

In the formula above, we assume the block has zero gravitational potential energy at point (B). The potential energy at point (A) is $U_i = mgh = mg|y_i - 0|\sin\theta$ *relative* to point (B), since point (A) is $h = |y_i - 0|\sin\theta$ metres higher than point (B).

Solving for v_B in this equation answers the first part of our question:

$$v_B = \sqrt{2g|y_i|\sin\theta}.$$

Now for the second part of the block's motion. The law of conservation of energy dictates that

$$K_i + U_{gi} + U_{si} = K_f + U_{gf} + U_{sf},$$

where i now refers to the moment (B), and f refers to the moment (C). Initially the spring is uncompressed, so $U_{si} = 0$. By the end of the motion, the spring is compressed by a total of $\Delta y = |y_f - 0|$[m], so its spring potential energy is $U_{sf} = \frac{1}{2}k|y_f|^2$. We choose the height of (C) as the reference potential energy; thus $U_{gf} = 0$. Since the difference in gravitational potential energy is $U_{gi} - U_{gf} = mgh = |y_f - 0|\sin\theta$, we can complete the entire energy equation:

$$\frac{1}{2}mv_B^2 + mg|y_f|\sin\theta + 0 = 0 + 0 + \frac{1}{2}k|y_f|^2.$$

Assuming the values of k and m are given, and knowing v_B from the first part of the question, we can solve for $|y_f|$ in the above equation.

To obtain the answer $|y_f|$ in terms of Δh, we'll use $\sum E_i = \sum E_f$ again, but this time i will refer to moment (A) and f to moment (C). The conservation of energy equation tells us $mg\Delta h = \frac{1}{2}k|y_f|$, from which we obtain $|y_f| = \frac{2mg\Delta h}{k}$.

Energy lost to friction

You place a block of mass 50[kg] on an incline. The force of friction between the block and the incline is 30[N]. The block slides for 200[m] down the incline. The incline's slope is $\theta = 30°$ making the block's total vertical displacement ~~200 sin 30 = 100~~$200\sin 30° = 100$[m]. What is the block's speed ~~as it reaches the bottom of the~~ after sliding for 200[m] down the incline?

This is a problem in which initial energies are converted into a combination of final energies and *lost* work:

$$\sum E_i = \sum E_f + W_{\text{lost}}.$$

The term W_{lost} represents energy lost due to friction.

A better way of describing this situation is that **a negative amount of work is done on the block**:

$$\sum E_i + \underbrace{W_{\text{done}}}_{\text{negative}} = \sum E_f.$$

The quantity W_{done} is negative because the friction force acts on the object in the opposite direction of the object's motion:

$$W_{\text{done}} = \vec{F} \cdot \vec{d} = \|\vec{F}_f\|\|\vec{d}\| \cos(180°) = -F_f\|\vec{d}\|,$$

where $\cancel{\vec{d}}$ $\|\vec{d}\|$ is the sliding distance of 200[m] over which the friction acts.

We substitute the value of W_{done} into the conservation of energy equation:

$$K_i + U_i + W_{\text{done}} = K_f + U_f,$$
$$0 + mgh + (-F_f|d|) = \frac{1}{2}mv_f^2 + 0.$$

Note we used the formula $mgh = U_i - U_f$ for the difference in gravitational potential energy.

Since we're told $F_f = 30$[N], we can calculate $W_{\text{done}} = W_{\text{friction}} = -30[\text{N}] \times 200[\text{m}] = -6000[\text{J}]$. Substituting all known values, we find

$$0 + 50 \times 9.81 \times 100 - 6000 = \frac{1}{2}(50)v_f^2 + 0,$$

which we can solve for v_f.

Discussion

It's useful to describe physical situations in terms of the energies involved. The law of conservation of energy allows us to use simple "energy accounting" principles to calculate the values of unknown quantities.

4.7 Uniform circular motion

This section covers the circular motion of objects. Circular motion differs from linear motion, and we'll need to learn new techniques and concepts specifically used to describe circular motion.

Imagine a rock of mass m attached to the end of a rope and swinging around in a horizontal circle. The rock flies through the air at a

Moment of inertia

An object's ~~momentum~~ moment of inertia describes how difficult it is to cause the object to rotate:

$$I = \{ \text{ how difficult it is to make an object turn } \}.$$

The calculation describing the moment of inertia accounts for the mass distribution of the object. An object with most of its mass close to its centre will have a smaller moment of inertia, whereas objects with masses far from their centres will have larger moments of inertia.

The formula for calculating the moment of inertia is

$$I = \sum m_i r_i^2 = \int_{\text{obj}} r^2 \; dm \qquad [\text{kg m}^2].$$

The contribution of each piece of the object's mass dm to the total moment of inertia is proportional to the squared distance of that piece from the object's centre, hence the units [kg m^2].

We rarely use the integral formula to calculate objects' moments of inertia. Most physics problems you'll be asked to solve will involve geometrical shapes for which the moment of inertia is given by simple formulas:

$$I_{\text{disk}} = \frac{1}{2}mR^2, \;\; I_{\text{ring}} = mR^2, \;\; I_{\text{sphere}} = \frac{2}{5}mR^2, \;\; I_{\text{sph.shell}} = \frac{2}{3}mR^2.$$

When you learn more about calculus (Chapter 5), you will be able to derive each of the above formulas on your own. For now, just try to remember the formulas for the inertia of the disk and the ring, as they are likely to show up in problems.

The quantity I plays the same role in the equations of angular motion as the mass m plays in the equations of linear motion.

Torques cause angular acceleration

Recall Newton's second law $F = ma$, which describes the amount of acceleration produced by a given force acting on an object. The angular analogue of Newton's second law is expressed as

$$\mathcal{T} = I\alpha. \tag{4.13}$$

This equation indicates that the angular acceleration produced by the torque $\mathcal{T}$ is inversely proportional to the object's moment of inertia. Torque is the cause of angular acceleration.

To solve this kinematics problem, we're looking for the angular acceleration produced by the brake. We can find it with the equation $\mathcal{T} = I\alpha$. We need to find $\mathcal{T}$ and I_{disk} and solve for α. The torque produced by the brake is calculated using the force-times-leverage formula: $\cancel{\mathcal{T} = F_{\perp} r = 60 \times 0.3 = 18}$ $\mathcal{T} = F_{\perp} r = 60 \times 0.3 = 18$[N~~-~~ m]. The moment of inertia of a disk is given by $I_{\text{disk}} = \frac{1}{2}mR^2 = \frac{1}{2}(20)(0.3)^2 = 0.9$[kg m^2]. Thus we have $\cancel{\alpha = 20}$ $\alpha = -\frac{18}{0.9} = -20$[rad/s^2]. Now we can use the UAM formula for angular velocity $\omega(t) = \alpha t + \omega_i$ and solve for the time when the object's motion will stop: $0 = \alpha t + \omega_i$. The disk will come to a stop after $\cancel{t = \omega_i/\alpha = 1}$ $t = -\omega_i/\alpha = 1$[s].

Combined motion

A pulley of radius R and moment of inertia I has a rope wound around it. At the end of the rope is attached a rock of mass m. What will be the angular acceleration of the pulley if we let the rock drop to the ground while unwinding the rope?

A force diagram of the rock tells us that $mg - T = ma_y$ (where $\hat{y}$ points downward). A torque diagram of the disk tells us that $TR = I\alpha$. Taking the product of R times the first equation and adding it to the second equation gives us

$$R(mg - T) + TR = Rma_y + I\alpha,$$

and after simplification we're left with

$$Rmg = Rma_y + I\alpha.$$

Additionally, since we know the rope forms a solid connection between the pulley and the rock, this means that the angular acceleration of the pulley is related to the linear acceleration of the rock: $R\alpha = a_y$. We can use this relationship between the variables a_y and α to obtain an equation with only one unknown. We substitute $R\alpha$ for a_y in the above equation to obtain

$$Rmg = Rm(R\alpha) + I\alpha = (R^2m + I)\alpha.$$

Solving for α we find

$$\alpha = \frac{Rmg}{R^2m + I}.$$

This answer makes sense intuitively. From the rotating disk's point of view, the cause of rotation is the torque produced by the falling mass, while the denominator represents the total moment of inertia for the mass-pulley system as a whole.

The vertical acceleration of the falling mass is obtained via $a_y = R\alpha$:

$$a_y = \frac{R^2mg}{R^2m + I} = \frac{mg}{m + \frac{I}{R^2}}.$$

seconds. Look carefully at the plot of the function $\cos(t)$. As t goes from $t = 0$ to $t = 2\pi$, the function $\cos(t)$ completes one full cycle. We say the *period* of $\cos(t)$ is $T = 2\pi$.

Input-scaling

If we want to describe a periodic motion with a different period, we can still use the cos function, but inside the cos function we must include a multiplier before the variable t. This multiplier describes the *angular frequency* and is denoted ω (*omega*). The input-scaled cos function

$$f(t) = \cos(\omega t)$$

has a period of $T = \frac{2\pi}{\omega}$.

Scaling the input of the cos function by the constant $\omega = \frac{2\pi}{T}$ produces a periodic function with period T. When you vary t from 0 to T, the quantity ωt goes from 0 to 2π, so the function $\cos(\omega t)$ completes one cycle. You shouldn't just take my word for this; try it yourself by ~~building~~ drawing a cos function with a period of 3 units.

The *frequency* of periodic motion describes the number of times per second the motion repeats. ~~A motion's~~ The frequency is equal to the inverse of ~~its~~ the period:

$$f = \frac{1}{T} = \frac{\omega}{2\pi} \text{ [Hz]}.$$

Frequency f and angular frequency ω are related by a factor of 2π. We need this multiplier since the natural cycle length of the cos function is 2π radians.

Output-scaling

We can scale the output of the cos function by a constant A, called the *amplitude*. The function

$$f(t) = A\cos(\omega t)$$

will oscillate between $-A$ and A.

Time-shifting

The motion described by the function $A\cos(\omega t)$ starts from its maximum value at $t = 0$. A mass-spring system described by the position function $x(t) = A\cos(\omega t)$ begins its motion with the spring maximally stretched $x_i \equiv x(0) = A$.

If we want to describe other starting positions for the motion, it may be necessary to introduce a *phase shift* inside the cos function:

$$x(t) = A\cos(\omega t + \phi).$$

The constant ϕ must be chosen so that at $t = 0$, the function $x(t)$ correctly describes the initial position of the system.

For example, if the harmonic motion starts from the system's centre $x_i \equiv x(0) = 0$ and initially moves in the positive direction, then the motion is described by the function $A\sin(\omega t)$. Or, since $\sin(\theta) = \cos(\theta - \frac{\pi}{2})$, we can describe the same motion in terms of a shifted cos function:

$$x(t) = A\cos\Big(\omega t - \frac{\pi}{2}\Big) = A\sin(\omega t).$$

Note, the function $x(t)$ correctly describes the system's initial position $x(0) = 0$.

By now, the meaning of all the parameters in the simple harmonic motion equation should be clear to you. The constant in front of the cos tells us the motion's amplitude A, and the multiplicative constant ω inside the cos is related to the motion's period/frequency: $\omega = \frac{2\pi}{T} = 2\pi f$. Finally, the additive constant ϕ is chosen depending on the initial conditions. ~~By now, the meaning of all the parameters in the simple harmonic motion equation should be clear to you.~~

Mass and spring

Okay, it's time to apply all this math to a physical system which exhibits simple harmonic motion: the mass-spring system.

An object of mass m ~~is~~ attached to a spring with spring constant k~~. If~~, when disturbed from rest, ~~this mass-spring system~~ will undergo simple harmonic motion with angular frequency ~~of~~

$$\omega = \sqrt{\frac{k}{m}}. \tag{4.16}$$

A stiff spring attached to a small mass will result in very rapid oscillations. A weak spring or a heavy mass will result in slow oscillations.

A typical exam question may tell you k and m and ask about the period T. If you remember the definition of T, you can easily calculate the answer:

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}.$$

Equations of motion

The general equations of motion for the mass-spring system are

$$x(t) = A\cos(\omega t + \phi), \tag{4.17}$$

$$v(t) = -A\omega\sin(\omega t + \phi), \tag{4.18}$$

$$a(t) = -A\omega^2\cos(\omega t + \phi). \tag{4.19}$$

The general shape of the function $x(t)$ is similar to that of a cos function. The *angular frequency* ω parameter is governed by the physical properties of the system. The parameters A and ϕ describe the specifics of the motion, namely, the *amplitude* of the oscillation and its starting position.

The function $v(t)$ is obtained, as usual, by taking the derivative of $x(t)$. The function $a(t)$ is obtained by taking the derivative of $v(t)$, which corresponds to the second derivative of $x(t)$. The velocity and acceleration are also periodic functions.

Motion parameters

The key motion parameter of SHM is how far the mass swings back and forth through the *centre position*. The amplitude A describes the maximum distance the mass will travel in the positive x-direction.

We can also find the maximum values of an object's velocity and acceleration by reading the coefficient located in front of sin and cos in the functions $v(t)$ and $a(t)$.

- The object's maximum velocity is $v_{\max} = A\omega$.
- The object's maximum acceleration is $a_{\max} = A\omega^2$.

The velocity function reaches its maximum as the object passes through the centre position. The acceleration is maximum when the spring is maximally stretched or compressed—these are the locations where the pull of the spring is the strongest.

You'll definitely be asked to solve for the quantities $v_{\max}$ and $a_{\max}$ in exercises and exams. This is an easy task if you remember the above formulas and you know the values of the amplitude A and the angular frequency ω. Note the term *amplitude* applies more generally to the constant in front of *any* sin or cos function. Thus, we say that $v_{\max} = A\omega$ is the amplitude of the velocity and $a_{\max} = A\omega^2$ is the amplitude of the acceleration.

Energy

The potential energy stored in a spring that is stretched or compressed by a length x is given by the formula $U_s = \frac{1}{2}kx^2$. Since we know $x(t)$, as $\omega = \frac{2\pi}{T}$, so the angular frequency for the pendulum is

$$\omega \equiv \frac{2\pi}{T} = \sqrt{\frac{g}{\ell}}\,. \tag{4.20}$$

Instead of describing the pendulum's position x with respect to ~~Cartesian coordinate~~ the xy-coordinate system, we describe its position in terms of the angle θ it makes with the vertical line that passes through the centre of the motion. The equations of motion are described in terms of *angular quantities*: the angular position θ, the angular velocity ω_θ, and the angular acceleration α_θ of the pendulum:

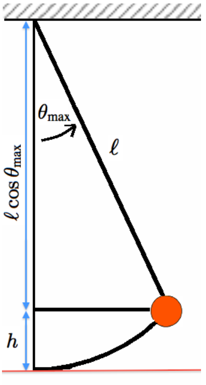
$$\begin{aligned}
\theta(t) &= \theta_{\max}\,\cos\left(\sqrt{\frac{g}{\ell}}t+\phi\right),\\
\omega_\theta(t) &= -\theta_{\max}\sqrt{\frac{g}{\ell}}\,\sin\left(\sqrt{\frac{g}{\ell}}t+\phi\right),\\
\alpha_\theta(t) &= -\theta_{\max}\frac{g}{\ell}\,\cos\left(\sqrt{\frac{g}{\ell}}t+\phi\right).
\end{aligned}$$

The angle $\theta_{\max}$ describes the maximum angle to ~~switch~~ which the pendulum swings. Notice the new variable name ω_θ we use for the pendulum's angular velocity $\omega_\theta(t) = \frac{d}{dt}(\theta(t))$. The angular velocity ω_θ of the pendulum should not be confused with the *angular frequency* $\omega = \sqrt{\frac{g}{\ell}}$ of the periodic motion, which is the constant inside the cos function.

Energy

A pendulum's motion is best understood by imagining how the energy in the system shifts between gravitational potential energy and kinetic energy.

The pendulum reaches its maximum potential energy when it swings sideways to reach angle $\theta_{\max}$. At this angle, the mass's vertical position is increased by a height h above the mass's lowest point. We can calculate h as follows:



$$h = \ell - \ell\cos\theta_{\max}.$$

The maximum gravitational potential energy of the mass is therefore

$$U_{g,\max} = mgh = mg\ell(1-\cos\theta_{\max}).$$

By the conservation of energy principle, the pendulum's maximum kinetic energy must equal its maximum gravitational potential energy:

$$mg\ell(1-\cos\theta_{\max}) = U_{g,\max} = K_{\max} = \frac{1}{2}mv_{\max}^2,$$

where $v_{\max} = \ell\omega_\theta$ is the linear velocity of the mass as it swings through the ~~vertical~~ centre position.

Explanations

It's worthwhile to understand where the simple harmonic motion equation comes from. In this subsection, we'll discuss how the equation $x(t) = A\cos(\omega t+\phi)$ is derived from Newton's second law $F = ma$ and the equation for the force of a spring $F_s = -kx$.

Trigonometric derivatives

The slope (derivative) of the function $\sin(t)$ varies between -1 and 1. The slope is largest when sin passes through the x-axis, and the slope is zero when the function reaches its maximum and minimum values. A careful examination of the graphs of the bare functions sin and cos reveals that the derivative of the function $\sin(t)$ is described by the function $\cos(t)$, and vice versa:

$$f(t) = \sin(t) \qquad \Rightarrow \qquad f'(t) = \cos(t),$$
$$f(t) = \cos(t) \qquad \Rightarrow \qquad f'(t) = -\sin(t).$$

When you learn more about calculus, you'll know how to find the derivative of any function you want; for now, you can take my word that the above two formulas are true.

The chain rule for derivatives (page 256) dictates that a composite function $f(g(x))$ has derivative $f'(g(x))\cdot g'(x)$. First we take the derivative of the outer function, then we multiply by the derivative of the inner function. We can find the derivative of the position function $x(t) = A\cos(\omega t+\phi)$ using the chain rule:

$$v(t) \equiv x'(t) = -A\sin(\omega t+\phi)\cdot\omega = -A\omega\sin(\omega t+\phi),$$

where the outer function is $f(x) = A\cos(x)$ with derivative $f'(x) = -A\sin(x)$, and the inner function is $g(x) = \omega x+\phi$ with derivative $g'(x) = \omega$.

The same reasoning is applied to obtain the second derivative:

$$a(t) \equiv \frac{d}{dt}\{v(t)\} = -A\omega^2\cos(\omega t+\phi) = -\omega^2 x(t).$$

Note the function $a(t) \equiv x''(t)$ has the same form as the function $x(t)$; the two functions differ only by the factor $-\omega^2$.

Derivation of the mass-spring SHM equation

You may be wondering where the equation $x(t) = A\cos(\omega t + \phi)$ comes from. This formula looks very different from the kinematics equation for linear motion $x(t) = x_i + v_i t + \frac{1}{2}at^2$, which we obtained starting with Newton's second law $F = ma$ and completing two steps of integration.

In this section, I've seemingly pulled the $x(t) = A\cos(\omega t + \phi)$ formula out of thin air, as if by revelation. Why did we suddenly start talking about cos functions and Greek letters with dubious names like "phase"? Are you ~~phased~~ fazed by all of this? When I was first learning about simple harmonic motion, I was totally ~~phased~~ fazed because I didn't see where the sin and cos were coming from.

The cos also comes from $F = ma$, but the story is a little more complicated this time. The force exerted by a spring is $F_s = -kx$. Since we assume the surface the mass slides along is frictionless, the only force acting on the mass is the force of the spring:

$$\sum F = F_s = ma \qquad \Rightarrow \qquad -kx = ma.$$

Recall that the acceleration function is the second derivative of the position function:

$$a(t) \equiv \frac{dv(t)}{dt} \equiv \frac{d^2x(t)}{dt^2} \equiv x''(t).$$

We can rewrite the equation $-kx = ma$ in terms of the function $x(t)$ and its second derivative:

$$\begin{aligned} -kx(t) &= m\frac{d^2x(t)}{dt^2} \\ -\frac{k}{m}x(t) &= \frac{d^2x(t)}{dt^2}, \end{aligned}$$

which can be rewritten as

$$0 = \frac{d^2x(t)}{dt^2} + \frac{k}{m}x(t).$$

This is called a *differential equation*. Instead of looking for an *unknown number* as in normal equations, in differential equations we are looking for an *unknown function* $x(t)$. We do not know what the function $x(t)$ is, but ~~we do know~~ the differential equation tells us one of its ~~properties—namely, that~~ properties: the second derivative of $x(t)$ ~~'s second derivative $x''(t)$~~ is equal to the negative of $x(t)$ multiplied by some constant.

To solve a differential equation ~~,~~ you must guess which function $x(t)$ satisfies this property. ~~There is an entire course called Differential~~

~~Equations, in which engineers and physicists learn how~~ Engineering and physics students must take a differential equations course to learn to do this guessing thing. Can you think of a function that, when multiplied by $\frac{k}{m}$, is equal to its second derivative?

Okay, I thought of one:

$$x_1(t) = A_1 \cos\left(\sqrt{\frac{k}{m}}t\right).$$

Come to think of it, there is also a second function that works:

$$x_2(t) = A_2 \sin\left(\sqrt{\frac{k}{m}}t\right).$$

You should try this for yourself. Verify that $x_1''(t) + \frac{k}{m}x_1(t) = 0$ and $x_2''(t) + \frac{k}{m}x_2(t) = 0$, which means these functions are *both* solutions to the differential equation $x''(t) + \frac{k}{m}x(t) = 0$. Since both $x_1(t)$ and $x_2(t)$ are solutions, ~~any combination of them~~ their sum must also be a solution:

$$x(t) = A_1 \cos(\omega t) + A_2 \sin(\omega t).$$

This is *kind of* the answer we're looking for: an expression that describes the object's position as a function of time. I say *kind of* because the solution we obtained is not specified as a cos function with amplitude A and a phase ϕ, but instead in terms of the coefficients A_1 and A_2, which describe the cos and sin components of the motion.

Lo and behold, using the trigonometric identity ~~from~~ $\cos(a+b) = \cos(a)\cos(b$ (see page 72$\cancel{\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)}$), we can rewrite the above expression for $x(t)$ as a time-shifted trigonometric function:

$$x(t) = A\cos(\omega t + \phi) = A_1 \cos(\omega t) + A_2 \sin(\omega t),$$

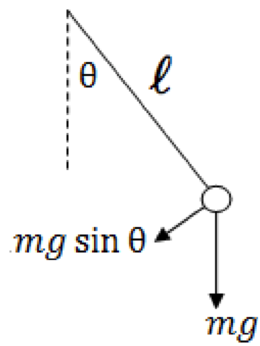
where $A_1 = A\cos(\phi)$ and $A_2 = -A\sin(\phi)$. The expression on the left is the preferred way of describing simple harmonic motion because the parameters A and ϕ correspond to observable aspects of the motion. If we know the coefficients A_1 and A_2, we can find the canonical parameters A and ϕ using $A = \sqrt{A_1^2 + A_2^2}$ and $\phi = \tan^{-1}(A_2/A_1)$.

Let's review one more time: we are looking for the equation of motion that predicts an object's position as a function of time $x(t)$. We can draw an analogy to a situation we've seen before. In linear kinematics, uniform accelerated motion with $a(t) = a$ is described by the equation $x(t) = x_i + v_i t + \frac{1}{2}at^2$ in terms of parameters x_i and v_i. Depending on the object's initial position and initial velocity, we obtain different trajectories. Simple harmonic motion with angular frequency ω is described by the equation $x(t) = A\cos(\omega t + \phi)$ in terms of the parameters A and ϕ. Depending on the values of the amplitude A and the phase ϕ, we obtain different simple harmonic motion trajectories.

Derivation of the pendulum SHM equation

To see how the simple harmonic motion equation for the pendulum is derived, we need to start from the torque equation $\mathcal{T} = I\alpha$.

The diagram illustrates how we can calculate the torque on the pendulum, which is caused by the force of gravity on the mass as a function of the ~~mass's~~ displacement angle θ. Recall the torque calculation only accounts for the $F_\perp$ component of any force, since this is the only part of the force that causes rotation:



$$\mathcal{T}_\theta = F_\perp \ell = -mg \sin\theta \ell.$$

The torque is negative because it acts in the opposite direction to the displacement angle θ.

Now we substitute this expression for $\mathcal{T}_\theta$ into the angular version of Newton's second law $\mathcal{T} = I\alpha$ to obtain

$$\begin{aligned} \mathcal{T} &= I\alpha \\ -mg\sin\theta(t)\ell &= m\ell^2 \frac{d^2\theta(t)}{dt^2} \\ -g\sin\theta(t) &= \ell \frac{d^2\theta(t)}{dt^2}\,. \end{aligned}$$

To continue with the derivation, we must make an approximation. When θ is a small angle, we can use the following approximation:

$$\sin(\theta) \;\approx\; \theta, \qquad\qquad \text{for } \theta \ll 1.$$

This is known as a *small angle approximation*. You'll see where it comes from later when you learn about Taylor series approximations to functions (page 337). For now, you can convince yourself of the above formula by zooming in near the origin on the graph of the function $\sin x$ until you realize $y = \sin(x)$ looks very much like $y = x$.

Using the small angle approximation $\sin\theta \approx \theta$, we rewrite the learned by studying oscillating systems where the energy is slowly dissipating. This is known as *damped harmonic motion*, for which the equation of motion looks like $x(t) = Ae^{-\gamma t}\cos(\omega t + \phi)$. This equation describes an oscillation with an amplitude that slowly decreases. The coefficient γ is known as the damping coefficient, and indicates how quickly the system's energy dissipates.

The concept of simple harmonic motion arises in many other areas of physics. When you learn about electric circuits, capacitors, and inductors, you'll run into equations of the form $v''(t) + \omega^2 v(t) = 0$, which indicates that a circuit's *voltage* is undergoing simple harmonic motion. Guess what—the same equation that describes the mechanical motion of the mass-spring system is used to describe the voltage in an oscillating circuit!

Links

[Plot of the simple harmonic motion using a can of spray-paint]
http://www.youtube.com/watch?v=p9uhmjbZn-c

[Slow motion movie clip of a mass-spring system]
http://bit.ly/QTRse3 [15 pendulums with different lengths]

http://www.youtube.com/watch?v=yVkdfJ9PkRQ

4.10 Conclusion

The fundamental purpose of mechanics is to predict the motion of objects using equations. In the beginning of the chapter, I claimed there are only 20 equations you need to know in order to solve any physics problem. Let us verify this claim and review the material we've covered.

Our goal was to find $x(t)$ for all times t. However, none of the equations of physics tell us $x(t)$ directly. Instead, we have Newton's second law $F = ma$, which tells us that the acceleration of the object $a(t)$ equals the *net force* acting on the object divided by the object's mass. To find $x(t)$ starting from $a(t)$, we use integration twice:

$$\frac{1}{m}\left(\sum \vec{F} \equiv \vec{F}_{\text{net}}\right) = a(t) \quad \xrightarrow{v_i + \int dt} \quad v(t) \quad \xrightarrow{x_i + \int dt} \quad x(t).$$

We studied kinematics in several different contexts. We originally looked at kinematics problems in one dimension, and derived the UAM and UVM equations. We also studied the problem of projectile motion by deconstructing it into two separate kinematics subproblems: one in the x-direction (UVM), and one in the y-direction (UAM). Later, we studied the circular motion of objects and stated

the pendulum's transversal displacement.
Sol: This is a simple harmonic motion question involving a pendulum. Begin by writing the general equation of motion for a pendulum: $\theta(t) = \theta_{\max}\cos(\omega t)$, where $\omega = \sqrt{g/\ell}$. Enter the walkway, which is moving to the left at velocity v. If we choose the $x = 0$ coordinate at a time when $\theta(t) = \theta_{\max}$, the pattern on the walkway can be described by the equation $y(x) = \ell\sin(\theta_{\max})\cos(kx)$, where $k = 2\pi/\lambda$, and λ tells us how long (measured as a distance in the x-direction) it takes for the pendulum to complete one cycle. One full swing of the bucket takes $T = 2\pi/\omega$[s]. In that time, the moving walkway will have moved a distance of vT metres. So one cycle in space (one wavelength) is $\lambda = vT = v2\pi/\omega$. We conclude that the equation of the paint on the moving sidewalk is $y(x) = \ell\sin(\theta_{\max})\cos((\omega/v)x)$. ~~Observe that the angular frequency parameter~~ $\cancel{\omega = \sqrt{g/\ell}}$ ~~does not depend on the mass of the pendulum; thus the change in mass as the paint leaks will not affect the pendulum's motion.~~

Links

[Physics exercises]
http://en.wikibooks.org/wiki/Physics_Exercises

[Lots of examples with solutions]
http://farside.ph.utexas.edu/teaching/301/lectures/lectures.html

The *derivative* function, denoted $f'(x)$ or $\frac{df}{dx}$, describes the *rate of change* of the function $f(x)$. For example, the constant function $f(x) = c$ has derivative $f'(x) = 0$ since the function $f(x)$ does not change at all.

The derivative function describes the *slope* of the graph of the function $f(x)$. The derivative of a line $f(x) = mx + b$ is $f'(x) = m$ since the slope of this line is equal to m. In general, the slope of a function is different at different values of x. For a given choice of input $x = x_0$, the value of the derivative function $f'(x_0)$ is equal to the slope of $f(x)$ as it passes through the point $(x_0, f(x_0))$.

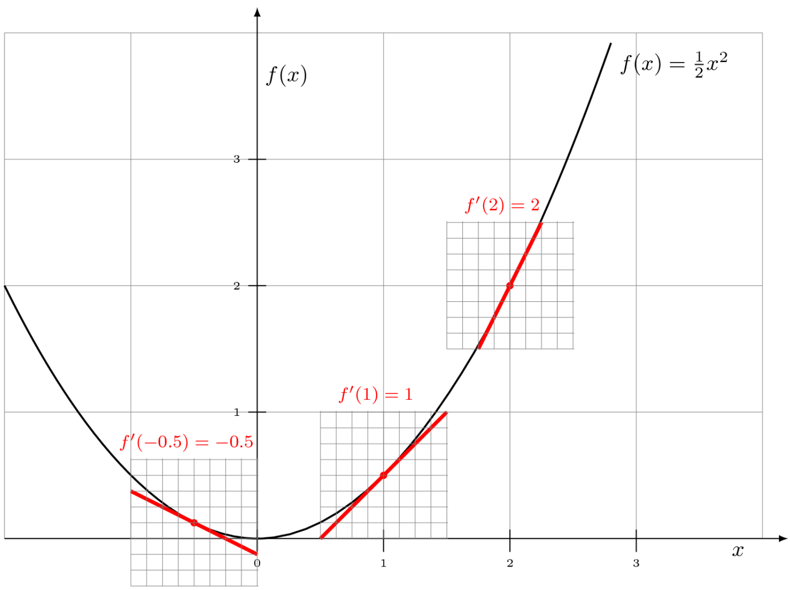


Figure 5.2: The diagram illustrates how to compute the derivative of the function $f(x) = \frac{1}{2}x^2$ at three different points on the graph of the function. To calculate the derivative of $f(x)$ at $x = 1$, we can "zoom in" near the point $\cancel{(1, 1)}$ $(1, \frac{1}{2})$ and draw a line that has the same slope as the function. We can then calculate the slope of the line using a rise-over-run calculation, aided by the mini coordinate system that is provided. The derivative calculations for $\cancel{x = -0.5}$ $x = -\frac{1}{2}$ and $x = 2$ are also shown. Note that the slope of the function is different for each value of x. What is the value of the derivative at $x = 0$? Can you find the general pattern?

The derivative function $f'(x)$ describes the slope of the graph of the function $f(x)$ for all inputs $x \in \mathbb{R}$. The derivative function is a function of the form $f' : \mathbb{R} \to \mathbb{R}$. In our study of mechanics, we learned about the position function $x(t)$ and the velocity function $v(t)$, which describe the motion of an object over time. The velocity is the deriva-

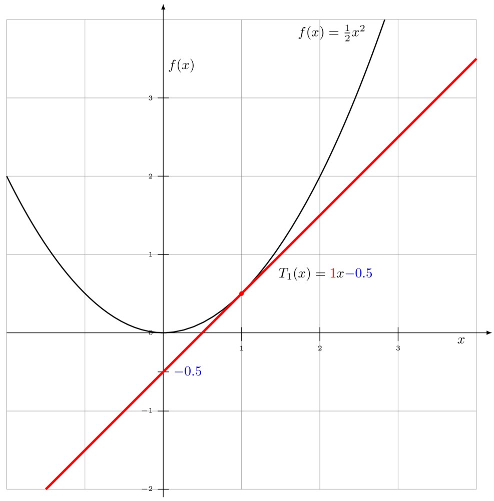


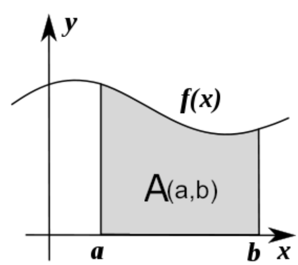
Figure 5.4: An illustration of the tangent line to the function $f(x) = \frac{1}{2}x^2$ at the point $\cancel{x = 1}$ $x_0 = 1$. The equation of the tangent line is $T_1(x) = 1x - 0.5$.

the input value x moves farther from x_0, the tangent becomes less accurate at approximating the function $f(x)$.

Integral calculus

The *integral* of $f(x)$ corresponds to the computation of the area under the graph of $f(x)$. The area under $f(x)$ between the points $x = a$ and $x = b$ is denoted as follows:

$$A(a, b) = \int_a^b f(x)\, dx.$$



The area $A(a, b)$ is bounded by the function $f(x)$ from above, by the x-axis from below, and by two vertical lines at $x = a$ and $x = b$. The points $x = a$ and $x = b$ are called the limits of integration. The $\int$ sign comes from the Latin word *summa*. The integral is the "sum" of the values of $f(x)$ between the two limits of integration.

The *integral function* $F(c)$ corresponds to the area calculation as a function of the upper limit of integration:

$$F(c) \equiv \int_0^c f(x)\, dx\,.$$

Applications of integration

Integral calculations have widespread applications to more areas of science than are practical to list here. Let's explore a few examples to gain a general idea of how integrals are applied in the real world.

Computing totals Integral calculations are needed every time we want to compute the total of some quantity that changes over time. If the quantity in question remains constant over time, we can multiply this quantity by the time to find the total quantity. For example, if your monthly rent is \$720, your annual rent is $\cancel{T = \$720 \times 12}$$R = \$720 \times 12$.

But what if your rent changes over time? Imagine a crazy landlord who demands you pay on a daily basis and changes the daily rent $r(t)$ each day. Some days rent is \$20/day, some days \$23/day, and some days he lets you stay for only \$15/day. In this situation, computing your annual rent involves the integral $\cancel{T = \int_0^{365} r(t)\, dt}$$R = \int_0^{365} r(t)\, dt$, which describes the calculation of the daily rate $r(t)$ times the duration of each day dt summed over all the days in the year.

Computing potentials In Section 4.6 we defined the notion of potential energy as the negative of the work done when moving an object against a conservative force. We studied two specific cases: gravitational potential energy $U_g(h) \equiv -\int_0^h \vec{F}_g \cdot d\vec{y} = mgh$, and spring potential energy $U_s(x) \equiv -\int_0^x \vec{F}_s(y) \cdot d\vec{y} = \frac{1}{2}kx^2$. Understanding integrals will allow you to solidify your understanding of the connection between each force $\vec{F}_?(x)$ and its associated potential energy $U_?(x)$.

Computing moments of inertia An object's moment of inertia describes how difficult it is to make the object turn. The moment of inertia is computed as the following integral:

$$I = \int_{\text{obj}} r^2 \, dm.$$

In the mechanics chapter, I asked you to memorize the formulas for $I_{\text{disk}} = \frac{1}{2}mR^2$ and $I_{\text{sphere}} = \frac{2}{5}mR^2$ because it was not yet time to explain the details of integral calculations. After learning about integrals, you'll be able to derive the formulas for I_{disk} and I_{sphere} on your own.

Solving differential equations One of the most important applications of integrals is their ability to "undo" the derivative operation.

Recall Newton's second law $F_{\text{net}}(t) = ma(t)$, which can also be written as

$$\frac{F_{\text{net}}(t)}{m} = a(t) = x''(t) = \frac{d}{dx}\left(\frac{d}{dx}x(t)\right).$$

In Chapter 2 we learned how to use integration to solve for $x(t)$ in special cases where the net force is constant $F_{\text{net}}(t) = F_{\text{net}}$. In this chapter, we'll revisit the procedure for finding $x(t)$, and learn how to calculate the motion of an object affected by an external force that varies over time $F_{\text{net}}(t)$.

Limits

The main new tool we'll use in our study of calculus is the notion of a *limit*. In calculus, we often use limits to describe what happens to mathematical expressions when one variable becomes very large, or alternately becomes very small.

For example, to describe a situation where a number n becomes bigger and bigger, we can say,

$$\lim_{n\to\infty} (\text{expression involving } n) \, .$$

This expression is read, "in the limit as n goes to infinity, expression involving n."

Another type of limit occurs when a small, positive number—for example $\delta > 0$, the Greek letter *delta*—becomes progressively smaller and smaller. The precise mathematical statement that describes what happens when the number δ tends to 0 is

$$\lim_{\delta\to 0} (\text{expression involving } \delta) \, ,$$

which is read as, "the limit as δ goes to zero, expression involving δ."

Derivative and integral operations are both defined in terms of limits, so understanding limits is essential for calculus. We'll explore limits in more detail and discuss their properties in Section 5.4.

Sequences

So far~~in this book, we~~, we've studied functions defined for real-valued inputs $x \in \mathbb{R}$. We can also study functions defined for natural number inputs $n \in \mathbb{N}$. These functions are called *sequences*.

A sequence is a function of the form $a : \mathbb{N} \to \mathbb{R}$. The sequence's input variable is usually denoted n or k, and it corresponds to the *index* or number in the sequence. We describe sequences either by

Techniques

The main mathematical question we'll study with series is the question of their convergence. We say a series $\sum a_n$ *converges* if the infinite sum $S_\infty \equiv \sum_{n\in\mathbb{N}} a_n$ equals some finite number $L \in \mathbb{R}$.

$$S_\infty = \sum_{n=0}^{\infty} a_n = L \quad \Rightarrow \quad \text{the series } \sum a_n \text{ converges.}$$

We call L the *limit* of the series $\sum a_n$.

If the infinite sum $S_\infty \equiv \sum_{n\in\mathbb{N}} a_n$ grows to infinity, we say the series $\sum a_n$ *diverges*.

$$S_\infty = \sum_{n=0}^{\infty} a_n = \pm\infty \quad \Rightarrow \quad \text{the series } \sum a_n \text{ diverges.}$$

The main series technique you need to learn is how to spot the differences between series that converge and series that diverge. You'll ~~soon learn to perform a number of~~ learn how to perform different convergence *tests* on the terms in the series, which will indicate whether the infinite sum converges or diverges.

Applications

Series are a powerful computational tool. We can use series to compute approximations to numbers and functions.

For example, the number e can be computed as the following series:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2\cdot 1} + \frac{1}{3\cdot 2} + \frac{1}{4\cdot 3\cdot 2} + \frac{1}{5\cdot 4\cdot 3\cdot 2} + \cdots.$$

The factorial operation $n!$ is the product of n times all integers smaller than n: $n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$. As we compute more terms from the series, our estimate of the number e becomes more accurate. The partial sum of the first six terms (as shown above) gives us an approximation of e that is accurate to three decimals. The partial sum of the first 12 terms gives us e to an accuracy of nine ~~digits.~~ decimals.

Another useful thing you can do with series is approximate functions by infinitely long polynomials. The *Taylor series* approximation for a function $f(x)$ is defined as the series

$$f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots.$$

Each term in the series is of the form $a_n = c_n x^n$, where c_n is a constant that depends on the function $f(x)$.

For example, the power series of $\sin(x)$ is

$$\sin(x) = \underbrace{\overbrace{x}^{T_1(x)} - \frac{x^3}{3!} + \frac{x^5}{5!}}_{T_5(x)} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots.$$

We can truncate the infinite series anywhere to obtain an approximation to the function. The function $T_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$ is the best approximation to the function $\sin(x)$ by a polynomial of degree 5. The equation of the tangent line $T_1(x)$ at $x = 0$ is a special case of the Taylor series approximation procedure, which approximates the function as a first-degree polynomial. We will continue the discussion on series, their properties, and their applications in Section 5.21.

If you haven't noticed yet from glancing at the examples so far, the common theme underpinning all the topics of calculus is the notion of *infinity*. We now turn our attention to the infinite.

5.3 Infinity

Working with infinitely small quantities and infinitely large quantities can be tricky business. It is important that you develop an intuitive understanding of these concepts as soon as possible. Like, now.

Infinitely large

The number ∞ is *really* large. How large? Larger than any number you can think of. Think of any number n. It is true that $n < \infty$. Now think of a bigger number N. It will still hold true that $N < \infty$. In fact, any finite number you can think of, no matter how large, will always be less than ∞.

Technically speaking, ∞ is not a number; infinity is a *process*. You can think of ∞ as the answer you obtain by starting from 0 and continuously adding 1 *forever*.

To see why $N < \infty$ for any finite number N, consider the following reasoning. When we add 1 to a number, we obtain a larger number. The operation $+1$ is equivalent to taking one unit step to the right on the number line. For any n, $n < n + 1$. To get to infinity we start from $n = 0$ and keep adding 1. After N ~~step~~steps, we'll arrive at $n = N$. But then we must continue adding 1 and obtain $N + 1$, $N + 2$, $N + 3$, and so on. Since adding 1 always creates a larger number, the following chain of inequalities is true:

$$N < N+1 < N+2 < N+3 < \cdots < \infty.$$

tried to verify his theory experimentally by placing himself in front of an arrow. A wrong argument about limits could get you killed!

Interlude

If the concept of infinity were a person, it would have several problematic character traits. Let's see what we know about infinity so far. The bit about the infinitely large shows signs of megalomania. There is enough of this whole "more, more, more" ~~stuff~~ mentality in the world already, so the last thing you want is someone like this as a friend. Conversely, the obsession with the infinitely small ϵ could be a sign of abnormal altruism: the willingness to give up all and leave less and less for oneself. You don't want someone *that* altruistic in your group. And that last part about how infinitely many numbers can fit in a finite interval of the number line sounds infinitely theoretical—definitely not someone to invite to a party. Let's learn about one redeeming, practical quality of the concept of infinity. Who knows, you might become friends after all.

Infinitely precise

A computer science (CS) student and a math student are chatting over lunch. The CS student recently learned how to write code that computes mathematical functions as infinitely long series:

$$f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2 \cdot 1} + \frac{x^3}{3 \cdot 2} + \frac{x^4}{4 \cdot 3 \cdot 2} + \cdots$$

She wants to tell her friend about her newly acquired powers.

The math student is also learning cool stuff about transcendental numbers. For example the number e can be defined as $e \equiv \lim_{n\to\infty}(1+\frac{1}{n})^n$, but can never be computed exactly—it can only be approximated.

"You know, math is *soooo* much better than CS," says the math student, baiting her friend into an argument about the relative merits of their fields of study.

"What? No way. I can do *anything* on a computer," replies the incredulous scholar of code.

"But can you find exact answers?" the mathematician asks. "Can you compute the number e *exactly*?"

"Sure," says the computer scientist, opening her laptop and typing in a few commands. "The answer is $e = 2.718281828459045$."

"That is not exact," the mathematician points out, "it is just an approximation."

compute $N = 19$ terms in the series:

$$e_{19} = \sum_{n=0}^{19} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{19!}\,.$$

The resulting approximation e_{19} is a number somewhere in the interval $(e - 10^{-15}, e + 10^{-15})$. We can also say the absolute value of the difference between e_{19} and the true value of e is smaller than ϵ: $|e_{19} - e| \leq 10^{-15}$.

When the mathematician asks for a precision of $\epsilon' = 10^{-25}$, the computer scientists takes $N = 26$ terms in the series to produce

$$e_{26} = \sum_{n=0}^{26} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{19!} + \cdots + \frac{1}{26!}\,,$$

which satisfies $|e_{26} - e| \leq \epsilon'$. In the third step, the mathematician demands a precision $\epsilon'' = 10^{-50}$, and the CS student computes $N = 42$ terms in the series, to produce an approximation satisfying $|e_{42} - e| \leq \epsilon''$. In principle, the game can continue indefinitely because the computer scientist has figured out a *process* for computing increasingly accurate approximations.

This scenario embodies precisely how mathematicians think about limits. It's a bit like a game: the ϵ,N-game. The object of the game is for the CS student to convince the mathematician she knows the number e. The mathematician chooses the precision ϵ. To prove she knows e to precision ϵ, the CS student computes the appropriate number of terms in the series such that her approximation e_N comes ϵ-close to the true answer $|e_N - e| < \epsilon$. If she can produce an approximation which satisfies $|e_N - e| < \epsilon$ **for all** $\epsilon > 0$, then the mathematician will be convinced.

Knowing the value of any finite approximation e_N, no matter how precise, does not constitute a mathematical proof that you can compute e. The mathematician is convinced because the computer scientist has found a *process* for computing approximations with arbitrary precision. In the words of the band Rage Against The Machine,

> “⟨~~EXPLETITIVE~~EXPLETIVE⟩ the G-rides,
> I want the machines that are making them.”

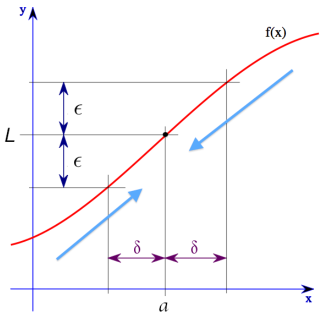
Calculus proofs are not about the approximations e_{19}, e_{26}, e_{42}, but about the machines that ~~make~~ are making them.

The scenarios presented in this section illustrate the need for a precise mathematical language for talking about infinitely large numbers, The limit of $f(x)$ when x approaches *from the left* is defined analogously,

$$\lim_{x\to a^-} f(x) = \lim_{\delta\to 0} f(a-\delta).$$

If both limits from the left and from the right of some number exist and are equal to each other, we can talk about the limit as $x \to a$ without specifying the direction of approach:

$$\lim_{x\to a} f(x) = \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x).$$



For the two-sided limit of a function to exist at a point, both the limit from the left and the limit from the right must converge to the same number. If the function $f(x)$ obeys, $f(a) = L$ and $\cancel{\lim_{x\to a} f(x) = L}$ $\lim_{x\to a} f(x) = L$, we say the function $f(x)$ is continuous at $x = a$.

Continuity

A function is said to be *continuous* if its graph looks like a smooth curve that doesn't make any sudden jumps and contains no gaps. If you can draw the graph of the function on a piece of paper without lifting your pen, the function is continuous.

A more mathematically precise way to define continuity is to say the function is equal to its limit for all x. We say a function $f(x)$ is *continuous* at a if the limit of f as $x \to a$ converges to $f(a)$:

$$\lim_{x\to a} f(x) = f(a).$$

Remember, the two-sided limit $\lim_{x\to a}$ requires both the left and the right limit to exist and to be equal. Thus, the definition of continuity implies the following equality:

$$\lim_{x\to a^-} f(x) = f(a) = \lim_{x\to a^+} f(x).$$

Consider the mathematical definition of continuity given in the equation above. Can you see how it connects to the intuitive idea of continuous functions as functions that can be drawn without lifting the pen?

Most functions we'll study in calculus are continuous, but not all functions are. Functions that are not defined for some value, as well as functions that make sudden jumps, are not continuous. For example, consider the function

$$f(x) = \frac{|x-3|}{x-3} = \begin{cases} 1 & \text{if } x \geq 3, \\ -1 & \text{if } x < 3. \end{cases}$$

This function is *continuous from the right* at the point $x = 3$, since $\lim_{x\to 3^+} f(x) = 1 = f(3)$. However, taking the limit from the left, we find $\lim_{x\to 3^-} f(x) = -1 \neq f(3)$. Therefore, the function is not continuous. The function $f(x)$ is continuous everywhere on the real line except at $x = 3$.

~~You are asked to calculate~~ We can calculate the limit $\lim_{x\to 5} \dfrac{2x+1}{x}$ ~~.~~ as follows:

$$\lim_{x\to 5} \frac{2x+1}{x} = \frac{2(5)+1}{5} = \frac{11}{5}.$$

There is nothing tricky going on ~~here—plug~~ here—we plug the number 5 into the equation, and voila. The function $f(x) = \frac{2x+1}{x}$ is continuous at the value $x = 5$, so the limit of the function as $x \to 5$ is equal to the value of the function $\lim_{x\to 5} f(x) = f(5)$.

Asymptotes

An *asymptote* of the function $f(x)$ is a line the function approaches but never touches. The word asymptote comes from the Greek *asumptotos*, which means "not falling together." For example, the line $y = 0$ (the x-axis) is an asymptote of the function $f(x) = \frac{1}{x}$ as x goes to infinity.

A *vertical asymptote* is a vertical line that the function approaches. For example, the function $f(x) = \frac{1}{3-x}$ has a vertical asymptote at $x = 3$. When the function approaches $x = 3$ from the left, the function increases to infinity:

$$\lim_{x\to 3^-} \frac{1}{3-x} = \infty.$$

The limit describes x taking on values like 2.9, 2.99, 2.999, and so on. The number in the denominator gets smaller and smaller, thus the fraction grows larger and larger. Note, the function is not defined at the exact value $x = 3$. Nevertheless, the above limit allows us to describe what happens to the function near that point.

Example 4 Find $\lim_{x\to 0} \frac{2x+1}{x}$.

Plugging $x = 0$ into the fraction yields a divide-by-zero error $\frac{2(0)+1}{0}$, so a more careful treatment is required.

First we'll consider the limit from the right $\lim_{x\to0+}\frac{2x+1}{x}$. We want to approach the value $x=0$ with small positive numbers. First we'll define a small positive number $\delta>0$, then choose $x=\delta$, and then compute the limit:

$$\lim_{\delta\to0}\frac{2(\delta)+1}{\delta}=2+\lim_{\delta\to0}\frac{1}{\delta}=2+\infty=\infty.$$

In this instance, we take it for granted that $\lim_{\delta\to0}\frac{1}{\delta}=\infty$. Intuitively, let's imagine what happens in the limit as δ approaches 0. When $\delta=10^{-3}$, the function value will be $\frac{1}{\delta}=10^3$. When $\delta=10^{-6}$, $\frac{1}{\delta}=10^6$. As $\delta\to0$, the expression $\frac{1}{\delta}$ becomes larger and larger all the way to infinity.

If we take the limit from the left, letting x take on small negative values, we obtain

$$\lim_{\delta\to0}f(-\delta)=\frac{2(-\delta)+1}{-\delta}=-\infty.$$

Since $\cancel{\lim_{x\to0+}f(x)}$ $\lim\limits_{x\to0^+}f(x)$ does not equal $\cancel{\lim_{x\to0-}f(x)}$ $\lim\limits_{x\to0^-}f(x)$, we say $\cancel{\lim_{x\to0}f(x)}$ $\lim\limits_{x\to0}f(x)$ does not exist.

Limits are fundamentally important for calculus. Indeed, the three main calculus topics we'll discuss in the remainder of this chapter are derivatives, integrals, and series—all of which are defined using limits.

Limits for derivatives

The formal definition of a function's derivative is expressed in terms of the rise-over-run formula for an infinitesimally short run:

$$f'(x)\;=\;\lim_{\text{run}\to0}\frac{\text{rise}}{\text{run}}\;=\;\lim_{\delta\to0}\frac{f(x+\delta)\;-\;f(x)}{x+\delta\;-\;x}.$$

We'll continue the discussion of this formula in Section 5.6.

Limit for integrals

One way to approximate the area under the curve $f(x)$ between $x=a$ and $x=b$ is to split the area into N little rectangles of width $\epsilon=\frac{b-a}{N}$ and height $f(x)$, and then calculate the sum of the areas of the rectangles:

$$A(a,b)\approx\underbrace{\epsilon f(a)+\epsilon f(a+\epsilon)+\epsilon f(a+2\epsilon)+\cdots+\epsilon f(b-\epsilon)}_{N\text{ terms}}.$$

$\sqrt{x} \equiv x^{\frac{1}{2}}$	$\frac{1}{2\sqrt{x}} \equiv \frac{1}{2}x^{-\frac{1}{2}}$
e^x	e^x
a^x	$a^x \ln(a)$
$\ln(x)$	$\frac{1}{x}$
$\log_a(x)$	$(x\ln(a))^{-1}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\sec^2(x) \equiv \cos^{-2}(x)$
$\sin^{-1}(x)$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1}(x)$	$\frac{-1}{\sqrt{1-x^2}}$
$\tan^{-1}(x)$	$\frac{1}{1+x^2}$
$\sinh(x)$	$\cosh(x)$
$\cosh(x)$	$\sinh(x)$

You can find a complete table of derivative formulas on page **??** in the back of the book.

5.8 Derivative rules

Taking derivatives is a simple task: find the appropriate formula in the table of derivative formulas and apply the formula to the specific problem at hand. Derivative tables come in handy, but they usually do not list formulas for *composite* functions. This section covers some important derivatives rules that will allow you to find derivatives of more complicated functions.

~~Linearity~~

Linearity

The derivative of a sum of two functions is the sum of the derivatives:

$$[f(x)+g(x)]' = f'(x)+g'(x),$$

and for any constant α, we have

$$[\alpha f(x)]' = \alpha f'(x).$$

The derivative of a linear combination of functions $\alpha f(x)+\beta g(x)$ is equal to the linear combination of the derivatives $\alpha f'(x)+\beta g'(x)$.

~~Product rule~~

Product rule

The derivative of a product of two functions is obtained as follows:

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x).$$

~~Quotient rule~~

Quotient rule

The quotient rule tells us how to obtain the derivative of a fraction of two functions:

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

~~Chain rule~~

Chain rule

If you encounter a situation that includes an inner function and an outer function, like $f(g(x))$, you can obtain the derivative by a two-step process:

$$[f(g(x))]' = f'(g(x))g'(x).$$

In the first step, leave the inner function $g(x)$ alone. Focus on taking the derivative of the outer function $f(x)$, and leave the expression $g(x)$ inside the f' expression. As the second step, multiply the resulting expression by the derivative of the *inner* function $g'(x)$.

The chain rule tells us the derivative of a composite function is calculated as the product of the derivative of the outer function and the derivative of the inner function.

Consider the following derivative calculation: ~~Exampl~~

$$\left[\sin(x^2)\right]' = \cos(x^2)\left[x^2\right]' = \cos(x^2)2x.$$

The chain rule also applies to functions of functions of functions $f(g(h(x)))$. To take the derivative, start from the outermost function and work your way toward x.

$$[f(g(h(x)))]' = f'(g(h(x)))g'(h(x))h'(x).$$

~~Now let's try taking another derivative~~

Let's compute the following derivative:

$$\left[\sin(\ln(x^3))\right]' = \cos(\ln(x^3))\left[\ln(x^3)\right]' = \cos(\ln(x^3))\frac{1}{x^3}\left[x^3\right]' = \cos(\ln(x^3))\frac{3}{x}.$$

Simple, right?

Examples

The above rules define *all* you need to know to take the derivative of any function, no matter how complicated. To convince you, I'll show you some examples of really hairy functions. Don't be scared by complexity: as long as you follow the rules, you'll find the right answer in the end.

Example Calculate the derivative of

$$f(x) = e^{x^2}.$$

We need the chain rule for this one:

$$f'(x) = e^{x^2}[x^2]' = e^{x^2}2x.$$

Example 2 Find the derivative of

$$f(x) = \sin(x)e^{x^2}.$$

We'll need the product rule for this one:

$$f'(x) = \cos(x)e^{x^2} + \sin(x)2xe^{x^2}.$$

Example 3 Compute the derivative of

$$f(x) = \sin(x)e^{x^2}\ln(x).$$

This situation again calls for the product rule, but this time we'll have three terms. For each term, take the derivative of one of the functions and multiply this derivative by the other two functions:

$$f'(x) = \cos(x)e^{x^2}\ln(x) + \sin(x)2xe^{x^2}\ln(x) + \sin(x)e^{x^2}\frac{1}{x}.$$

Example 4 Take the derivative of

$$f(x) = \sin(\cos(\tan(x))).$$

We need a triple chain rule for this one:

$$\begin{aligned} f'(x) &= \cos(\cos(\tan(x)))\,[\cos(\tan(x))]' \\ &= -\cos(\cos(\tan(x)))\sin(\tan(x))\,[\tan(x)]' \\ &= -\cos(\cos(\tan(x)))\sin(\tan(x))\sec^2(x). \end{aligned}$$

The trick is to define a new quantity

$$\Delta = g(x+\delta) - g(x),$$

and then substitute $g(x+\delta) = g(x)+\Delta$ into the derivative expression:

$$[f(g(x))]' = \lim_{\delta\to 0} \frac{f(g(x)+\Delta) - f(g(x))}{\delta}.$$

This is starting to look more like a derivative formula, but the quantity added in the input is different from the quantity by which we divide. To fix this, we can multiply and divide by Δ and rearrange the expression to obtain

$$\lim_{\delta\to 0} \frac{f(g(x)+\Delta) - f(g(x))}{\delta}\frac{\Delta}{\Delta} = \lim_{\delta\to 0} \frac{f(g(x)+\Delta) - f(g(x))}{\Delta}\frac{\Delta}{\delta}.$$

Now use the definition of the quantity Δ and rearrange the fraction:

$$[f(g(x))]' = \lim_{\delta\to 0} \frac{f(g(x)+\Delta) - f(g(x))}{\Delta}\frac{g(x+\delta)-g(x)}{\delta}.$$

This looks a lot like $f'(g(x))g'(x)$, and in fact, it is. Taking the limit $\delta \to 0$ implies that the quantity $\Delta(\delta) \to 0$. This is because the function $g(x)$ is continuous: $\lim_{\delta\to 0}[g(x+\delta) - g(x)] = 0$. Taking a derivative by using the quantity Δ is just as good as using δ. Thus, we've shown that

$$[f(g(x))]' = f'(g(x))g'(x).$$

Alternate notation

The presence of so many primes and brackets can make the expressions above difficult to read. As an alternative, we sometimes use another notation for derivatives. The three rules of derivatives in the alternate notation are written as follows:

- Linearity: $\frac{d}{dx}(\alpha f(x) + \beta g(x)) = \alpha\frac{df}{dx} + \beta\frac{dg}{dx}$
- Product rule: $\frac{d}{dx}(f(x)g(x)) = \frac{df}{dx}g(x) + f(x)\frac{dg}{dx}$
- Chain rule: $\frac{d}{dx}\left(f(g(x))\right) = \frac{df}{dg}\frac{dg}{dx}$

Some authors prefer the notation $\frac{df}{dx}$ for the derivative of the function $f(x)$, because it is more evocative of a rise-over-run calculation.

Optimization: the killer app of calculus

Knowing your derivatives will allow you to *optimize* any function—a crucial calculus skill. Suppose you can choose the input of $f(x)$ and you want to pick the *best* value of x. The best value usually means the *maximum* value (if the function measures something desirable like profits) or the *minimum* value (if the function describes something undesirable like costs). We'll discuss the *optimization algorithm* in more detail in the next section, but first let us look at an example.

~~Crime TV~~

Crime TV

A calculus teacher turned screenwriter is working on the pilot episode for a new TV series. Here is the story he has written so far.

The local drug boss has recently been running into problems as police are intercepting his dealers on the street. The more drugs he sells, the more money he makes; but if he sells too much, police arrests will increase and he'll lose money. Fed up with this situation, he decides to find the *optimal* amount of drugs to release on the streets: as much as possible, but not enough to trigger the police raids. One day he tells his brothers and sisters in crime to leave the room and picks up a pencil and a piece of paper to do some calculus.

If x is the amount of drugs he releases on the street every day, then the amount of money he makes is given by the function

$$f(x) = 3000xe^{-0.25x},$$

where the linear part $3000x$ represents his profits with no police involvement and the $e^{-0.25x}$ represents the effects of the police stepping up their actions as more drugs are released.

Looking at the function, the drug boss asks, "What is the value of x which will give me the most profit from my criminal dealings?" Stated mathematically, he is asking,

$$\underset{x}{\operatorname{argmax}}\ 3000xe^{-0.25x} = ?$$

which means "find the value of the argument x that gives the *maximum* value of $f(x)$."

Remembering a conversation with a crooked financial analyst he met in prison, the drug boss recalls the steps required to find the maximum of a function. First he must take the function's derivative. Because the function is a product of two functions, he applies the

- *saddle point*: a place where $f'(x) = 0$ at a point that is neither a max nor a min. For example, the function ~~$f(x) = x^5$~~ $f(x) = x^3$ has a saddle point at $x = 0$.

Suppose some function $f(x)$ has a global maximum at x^*, and the value of that maximum is $f(x^*) = M$. The following mathematical notations apply:

- $\text{argmax}_x \ f(x) = x^*$: the location (the *argument* of the function) where the maximum occurs
- $\max_x \ f(x) = M$: the maximum value

Algorithm for finding extrema

Input: a function $f(x)$ and a constraint region $C = [x_i, x_f]$
Output: the location and value of all maxima and minima of $f(x)$

Follow this algorithm step-by-step to find the extrema of a function:

1. First, *look* at $f(x)$. If you can plot it, plot it. If not, try to imagine what the function looks like.

2. Find the derivative $f'(x)$.

3. Solve the equation $f'(x) = 0$. Usually, there will be multiple solutions. Make a list of them. We'll call this the list of *candidates*.

4. For each candidate x^* on the list, check to see whether it is a max, a min, or a saddle point:

   - If $f'(x^* - 0.1)$ is positive and $f'(x^* + 0.1)$ is negative, then the point x^* is a max. The function goes up, flattens at x^*, then goes down after x^*. Therefore, x^* must be a peak.
   - If $f'(x^* - 0.1)$ is negative and $f'(x^* + 0.1)$ is positive, the point x^* is a min. The function goes down, flattens, then goes up, so the point must be a minimum.
   - If $f'(x^* - 0.1)$ and $f'(x^* + 0.1)$ have the same sign, the point x^* is a saddle point. Remove it from the list of candidates.

5. Now go through the list one more time and reject all candidates x^* that do not satisfy the constraints C. In other words, if $x \in [x_i, x_f]$, the candidate stays; but if $x \notin [x_i, x_f]$, we remove it since this solution is not *feasible*. Returning to the alcohol consumption example, if you have a candidate solution that says you should drink 5[L] of booze, you must reject it because otherwise you would die.

   (a) For $x = -2$, we check $f'(-2.1) = 4(-2.1)(-2.1-2)(-2.1+2) < 0$ and $f'(-1.9) = 4(-1.9)(-1.9-2)(-1.9+2) > 0$ to conclude $x = -2$ must be a minimum.
   (b) For $x = 0$ we try $f'(-0.1) = 4(-0.1)(-0.1-2)(-0.1+2) > 0$ and $f'(0.1) = 4(0.1)(0.1-2)(0.1+2) < 0$, which reveals we have a maximum at $x = 0$.
   (c) For $x = 2$, we check $f'(1.9) = 4(1.9)(1.9-2)(1.9+2) < 0$ and $f'(2.1) = 4(2.1)(2.1-2)(2.1+2) > 0$, so $x = 2$ must be a minimum.

5. We don't have any constraints, so all of the above candidates make the cut.

6. We add the two constraint boundaries $-\infty$ and ∞ to the list of candidates. At this point, our final shortlist of candidates contains $\{x = -\infty, x = -2, x = 0, x = 2, x = \infty\}$.

7. We now evaluate the function $f(x)$ for each of the values to obtain location-value pairs $(x, f(x))$, like so: $\{(-\infty, \infty)$, $(-2, 340)$, $(0, 356)$, $(2, 340)$, $(\infty, \infty)\}$. Note that $f(\infty) = \lim_{x\to\infty} f(x) = \infty^4 - 8\infty^2 + 356 = \infty$ and the same is true for $f(-\infty) = \infty$.

We are done. The function has no global maximum since it increases to infinity. It has a local maximum at $x = 0$ with value 356. It also has two global minima at $x = -2$ and $x = 2$, both of which have value 340. Thank you, come again.

Alternate algorithm

Instead of checking nearby points to the left and right of each critical point, we can modify the algorithm with an alternate Step 4 known as the *second derivative test*. Recall the second derivative tells us the function's *curvature*. If the second derivative is positive at a critical point x^*, then the point x^* must be a minimum. If, on the other hand, the second derivative at a critical point is negative, the function must be maximum at x^*. If the second derivative is zero, the test is inconclusive.

Alternate Step 4

- ~~Check~~ For each candidate x^* ~~to determine if~~, see if it is a max, a min, or a saddle point~~.~~:
  - ▷ If $f''(x^*) < 0$ then x^* is a max.
  - ▷ If $f''(x^*) > 0$ then x^* is a min.

- ▷ If $f''(x^*) = 0$ ~~then~~, the second derivative test fails. We must revert back to checking nearby values $f'(x^* - \delta)$ and $f'(x^* + \delta)$ to determine if x^* is a max, a min, or a saddle point.

Limitations

The optimization algorithm above applies to *differentiable* functions of a single variable. Not all functions are differentiable. Functions with sharp corners, such as the absolute value function $|x|$, are not differentiable everywhere, and therefore won't work with the algorithm above. Functions with jumps in them, like the Heaviside step function, are not continuous and therefore not differentiable—the algorithm cannot be used on them either.

We can generalize the optimization procedure, which help us optimize functions of multiple variables $f(x, y)$. You'll learn how to do this in the course *multivariable calculus*. The optimization techniques will be similar to the steps above, but with more variables and more intricate constraint regions.

At last, I want to comment on the fact that you can only maximize *one* function. Say the drug boss from the TV series wanted to maximize his funds $f(x)$ *and* his gangster street cred $g(x)$. This is not a well-posed problem; either you maximize $f(x)$ or you maximize $g(x)$, but you can't do both. There is no reason why a single x would give the highest value for both $f(x)$ and $g(x)$. If both functions are important to you, you can make a new function that combines the original two $F(x) = f(x) + g(x)$ and maximize $F(x)$. If gangster street cred is three times more important to you than funds, you could optimize $F(x) = f(x) + 3g(x)$, but it is mathematically and logically impossible to maximize two things at the same time.

Exercises

The function $f(x) = x^3 - 2x^2 + x$ has a local maximum on the interval $x \in [0, 1]$. Find where this maximum occurs, and find the value of f at that point.
Ans: $\left(\frac{1}{3}, \frac{4}{27}\right)$.

5.11 Implicit differentiation

Thus far, we've discussed how to compute derivatives of functions $f(x)$. When we identify the function's output with the variable y, we can write $y(x) = f(x)$, which shows the variable y depends on x through the function $f(x)$. The slope of this function is calculated

Example Compute :

$$\sum_{n=0}^{\infty} \frac{1}{N+1}\left(\frac{N}{N+1}\right)^n = \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} = \frac{\frac{1}{N+1}}{1-\frac{N}{N+1}} = 1.$$

Computational example Compute $\sin(40°)$ to 15 decimal places. The Maclaurin series of $\sin(x)$ is

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

To calculate the sine of 40 degrees, we compute the sum of the series with x replaced by 40 degrees (expressed in radians). In theory, we need to sum *infinitely* many terms in the series, but in practice we only need to sum the first 8 terms in the series to obtain an accuracy of 15 digits after the decimal. In other words, the series converges very quickly.

Let's use the computer algebra system at `live.sympy.org` to compute the first few terms in the series to see what is going on.

First, we define the n^{th} term:

$$a_n(x) = \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

```
>>> def axn_sin(x,n):
        return (-1.0)**n * x**(2*n+1) / factorial(2*n+1)
```

Next we convert $40°$ to radians:

```
>>> forty = (40*pi/180).evalf()
    0.698131700797732                # 40 degrees in radians
```

~~These are the~~ Let's look at the list of the first 10 coefficients in the series:

```
>>> [ axn_sin(forty,n) for n in range(0,10) ]
    [ 0.69813170079773179,           # a_0
     -0.056710153964883062,          # a_1
      0.0013819920621191727,         # a_2
     -1.6037289757274478e-05,        # a_3
      1.0856084058295026e-07,        # a_4
     -4.8101245792792 79e-10,        # a_5
      1.5028144059670851e-12,        # a_6
     -3.4878738801065803e-15,        # a_7
      6.2498067170560129e-18,        # a_8
     -8.9066666494280343e-21 ]       # a_9
```

To compute $\sin(40°)$, we sum together all the terms:

```
>>> sum( [ axn_sin(forty,n) for n in range(0,10) ] )
    0.642787609686539       # the Taylor series approximation

>>> sin(forty).evalf()
    0.642787609686539       # the true value of sin(40)
```

Note the first 8 terms of the series would have been sufficient to obtain an approximation to 15 decimals since the terms a_8 and a_9 are much smaller than 10^{-15}.

Discussion

You can think of the Taylor series as containing the "similarity coefficients" between $f(x)$ and the different powers of x. We choose the terms in the Taylor series of $f(x)$ to ensure the series approximation has the same n^{th} derivative as the function $f(x)$. For a Maclaurin series, the similarity between $f(x)$ and its power series representation is measured at $x = 0$, so the coefficients are chosen as $c_n = \frac{f^{(n)}(0)}{n!}$. The more general Taylor series allows us to build an approximation to $f(x)$ at any point $x = a$, so the similarity coefficients are calculated to match the derivatives at that point: $c_n = \frac{f^{(n)}(a)}{n!}$.

Another way of looking at the Maclaurin series is to imagine it is a kind of X-ray picture for each function $f(x)$. The zero$^{\text{th}}$ coefficient c_0 in the Maclaurin series tells you how much of the constant function is in $f(x)$. The first coefficient, c_1, tells you how much of the linear function x is in f; the coefficient c_2 tells you about the x^2 contents of f, and so on.

Now get ready for some crazy shit. I want you to go back to page 338 and take a careful look at the Maclaurin series of e^x, $\sin(x)$, and $\cos(x)$. As you will observe, it's as if e^x *contains* both $\sin(x)$ and $\cos(x)$, the only difference being the presence of the alternating negative signs. How about that? Do you remember Euler's formula $e^{ix} = \cos x + i \sin x$? Verify Euler's formula (page 141) by substituting ix into the power series for e^x.

Another interesting equation to think about in terms of series is $e^x = \cosh x + \sinh x$.

Links

[Animation showing Taylor series approximations to $\sin(x)$]
http://mathforum.org/mathimages/index.php/Taylor_Series

End matter

Conclusion

We managed to cover a lot of ground, explaining many topics and concepts in a relatively small textbook. We reviewed high school math and learned about mechanics and calculus. Above all, we examined math and physics material in an integrated manner.

If you liked or hated this book, be sure to send me feedback. Feedback is crucial so I know how to adjust the writing, the content, and the attitude of the book for future learners of math. Please take the time to drop me a line and let me know what you thought. You can reach me by email at `ivan.savov@gmail.com`.

~~I have a followup physics book on electricity and magnetism in the works. Another title on linear algebra is also nearly complete. To stay informed about new titles, follow me on the twitter , the facebook , or~~ If you want to learn about other books in the No bullshit guide series and hear about the technology we're using at `Minireference Publishing` to take over the textbook industry, check out the company blog at `minireference.com/blog/`. You can also find us on the twitter `@minireference` and on the facebook `fb.me/noBSguide`.

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This book would not have been possible without the support and encouragement of the people around me. I am fortunate to have grown up surrounded by good people who knew the value of math and encouraged me in my studies and with this project. In this section, I want to *big up* all the people who deserve it.

First and foremost in this list are my parents from whom I have learned many things, and who have supported me throughout my life.

Next in line are all my teachers. I thank my CEGEP teachers: Karnig Bedrossian from whom I learned calculus, Paul Kenton from whom I learned how to think about physics in a chill manner, and

Benoit Larose who taught me that more dimensions does not mean things get more complicated. I thank Kohur Gowrisankaran, Frank Ferrie, Mourad El-Gamal, and Ioannis Psaromiligkos for their teaching during my engineering days, and Guy Moore and Zaven Altounian for teaching me advanced physics topics. Among all my teachers, I owe the most to Patrick Hayden whose teaching methods have always inspired me. From him, I learned that by defining things clearly, you can *trick* students into learning advanced topics, and even make it seem that the results are obvious! Thanks go out to all my research collaborators and friends: David Avis, Arlo Breault, Juan Pablo Di Lelle, Omar Fawzi, Adriano Ferrari, Igor Khavkine, Doina Precup, Andie Sigler, and Mark M. Wilde. Thank you all for teaching me a great many things!

Preparing this book took many years and a lot of effort ~~.~~ by many people. I want to thank Afton Lewis, Oleg Zhoglo, and Alexandra Foty for helping me proofread v2.0 of the book, and all the ~~readers of~~ v3 and v4.0 ~~who~~ readers who reported typos and suggested clarifications. Thank you all for your comments and feedback~~.~~! Georger Araujo deserves a particular mention as his meticulous reading of the text led to the correction of many technical mistakes. Above all, I want to thank my editor Sandy Gordon, who helped me ~~prepare v4.0 of the book, which is a substantial improvement over previous versions~~substantially improve the writing in the book. Her expertise with the English language and her advice on style and content have been absolutely invaluable.

Last but not least, I want to thank all my students for their endless questions and demands for explanations. If I have developed any skill for explaining things, I owe it to them.

Further reading

You have reached the end of this book, but you~~are~~’re only at the beginning of the journey of scientific discovery. There are a lot of cool things left for you to learn about. Below are some recommendation of subjects you might find interesting.

Electricity and Magnetism

Electrostatics is the study of the electric force $\vec{F}_e$ and the associated electric potential U_e. Here, you will also learn about the electric field $\vec{E}$ and electric potential V.

Magnetism is the study of the magnetic force $\vec{F}_b$ and the magnetic field $\vec{B}$, which are caused by electric currents flowing through wires.

General mathematics

Mathematics is a very broad field. There are all kinds of topics to learn about; some of them are fun, some of them are useful, some of them are boring, and some of them ~~which are~~ are totally mind expanding.

I recently discovered a book that covers many math topics of general interest and serves as a great overview of the many areas of mathematics. I highly recommend you take a look at this book.

[BOOK] Richard Elwes. *Mathematics 1001: Absolutely Everything That Matters About Mathematics in 1001 Bite-Sized Explanations*, Firefly Books, 2010, ISBN 1554077192.

General physics

If you want to learn more about physics, I highly recommend the Feynman lectures on physics. This three-tome collection covers all of undergraduate physics with countless links to more advanced topics. ~~While on the Feynman note, I want to also recommend his other book with life stories.~~

[BOOK] Richard P. Feynman, Robert B. Leighton, Matthew Sands. *The Feynman Lectures on Physics including Feynman's Tips on Physics: The Definitive and Extended Edition*, Addison Wesley, 2005, ISBN 0805390456.

While on the Feynman note, I want to also recommend his other book about life.

[BOOK] Richard P. Feynman. *Surely You're Joking, Mr. Feynman! (Adventures of a Curious Character)*, W. W. Norton & Company, 1997, ISBN 0393316041.

Lagrangian mechanics

In this book we learned about *Newtonian mechanics*, that is, mechanics starting from Newton's laws. There is a much more general framework known as Lagrangian mechanics which can be used to analyze more complex mechanical systems. The following is an excellent book on the subject.

[BOOK] Herbert Goldstein, Charles P. Poole Jr., John L. Safko. *Classical Mechanics*, Addison-Wesley, Third edition, 2001, ISBN 0201657023.

Final words

Throughout this book, I strived to equip you with the tools you'll need to make your future science studies enjoyable and pain free. Remember to always take it easy. Play with math and never take things too seriously. Grades don't matter. Big paycheques don't matter. Never settle for a boring job just because it pays well. Try to work only on projects you care about.

I want you to be confident in your ability to handle math, physics, and the other complicated stuff life will throw at you. You have the tools to do anything you want; choose your own adventure. And if the big banks come-a-knocking one day with a big paycheque trying to bribe you into applying your analytical skills to their avaricious schemes, ~~you can~~ send them-a-walking.

Appendix A

Constants, units, and conversion ratios

~~It~~ In this appendix you will find a number of tables of useful information ~~which~~ that you might need when solving math and physics problems.

Fundamental constants of Nature

Many of the equations of physics include constants as parameters of the equation. For example, Newton's law of gravitation says that the force of gravity between two objects of mass M and m separated by a distance r is $F_g = \frac{GMm}{r^2}$, where G is Newton's gravitational constant.

Symbol	Value	Units	Name
G	$6.673\,84 \times 10^{-11}$	$\text{m}^3\,\text{kg}^{-1}\text{s}^{-2}$	gravitational constant
g	$9.806\,65 \approx 9.81$	m s^{-2}	Earth free-fall acceleration
m_p	$1.672\,621 \times 10^{-27}$	kg	proton mass
m_e	$9.109\,382 \times 10^{-31}$	kg	electron mass
N_A	$6.022\,141 \times 10^{23}$	mol^{-1}	Avogadro's number
k_B	$1.380\,648 \times 10^{-23}$	J K^{-1}	Boltzmann's constant
R	$8.314\,462\,1$	$\text{J K}^{-1}\,\text{mol}^{-1}$	gas constant $R = N_\text{A} k_\text{B}$
μ_0	$1.256\,637 \times 10^{-6}$	N A^{-2}	permeability of free space
ε_0	$8.854\,187 \times 10^{-12}$	F m^{-1}	permittivity of free space
c	$299\,792\,458$	m s^{-1}	speed of light $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$
e	$1.602\,176 \times 10^{-19}$	C	elementary charge
h	$6.626\,069 \times 10^{-34}$	J s	Planck's constant

Mechanics formulas

Forces:

$$W = F_g = \frac{GMm}{r^2} = gm, \quad F_s = -kx, \quad F_{fs} \leq \mu_s N, \quad F_{fk} = \mu_k N$$

Newton's three laws:

$$\text{if no } \vec{F}_{\text{ext}}, \text{ then } \vec{v}_i = \vec{v}_f \tag{1}$$
$$\vec{F}_{\text{net}} = m\vec{a} \tag{2}$$
$$\text{if } \vec{F}_{12}, \text{ then } \exists \vec{F}_{21} = -\vec{F}_{12} \tag{3}$$

Uniform acceleration motion (UAM):

$$a(t) = a \tag{4}$$
$$v(t) = at + v_i \tag{5}$$
$$x(t) = \tfrac{1}{2}at^2 + v_i t + x_i \tag{6}$$
$$v_f^2 = v_i^2 + 2a\Delta x \tag{7}$$

Momentum:

$$\vec{p} = m\vec{v} \tag{8}$$

Energy and work:

$$K = \tfrac{1}{2}mv^2, \quad U_g = mgh, \quad U_s = \tfrac{1}{2}kx^2, \quad K_r = \tfrac{1}{2}I\omega^2, \quad W = \vec{F} \cdot \vec{d} \tag{9}$$

Conservation laws:

$$\sum \vec{p}_{\text{in}} = \sum \vec{p}_{\text{out}} \tag{10}$$
$$L_{\text{in}} = L_{\text{out}} \tag{11}$$
$$\sum E_{\text{in}} + W_{\text{in}} = \sum E_{\text{out}} + W_{\text{out}} \tag{12}$$

Circular motion (radial acceleration and radial force):

$$a_r = \frac{v_t^2}{R}, \qquad \vec{F}_r = ma_r\hat{r} \tag{13}$$

Angular motion:

$$F = ma \quad \Rightarrow \quad \mathcal{T} = I\alpha \tag{14}$$
$$a(t), v(t), x(t) \quad \Rightarrow \quad \alpha(t), \omega(t), \theta(t) \tag{15}$$
$$\vec{p} = m\vec{v} \quad \Rightarrow \quad L = I\omega \tag{16}$$
$$K = \tfrac{1}{2}mv^2 \quad \Rightarrow \quad K_r = \tfrac{1}{2}I\omega^2 \tag{17}$$

SHM with $\omega = \sqrt{\frac{k}{m}}$ (mass-spring system) or $\omega = \sqrt{\frac{g}{\ell}}$ (pendulum):

$$x(t) = A\cos(\omega t + \phi) \tag{18}$$
$$v(t) = -A\omega\sin(\omega t + \phi) \tag{19}$$
$$a(t) = -A\omega^2\cos(\omega t + \phi) \tag{20}$$