Common quadratic forms

Let's look at some common variations of quadratic expressions you might encounter when doing algebra calculations.

The quadratic expression $x^2 - p^2$ is called a *difference of squares*, and it can be obtained by multiplying the factors $(x+p)$ and $(x-p)$:

$$(x+p)(x-p) = x^2 \cancel{-xp} \cancel{+px} - p^2 = x^2 - p^2.$$

There's no linear term because the $-xp$ term cancels the $+px$ term. Any time you see an expression like $x^2 - p^2$, you can know it comes from a product of the form $(x+p)(x-p)$.

A *perfect square* is a quadratic expression that can be written as the product of repeated factors $(x+p)$:

$$x^2 + 2px + p^2 = (x+p)(x+p) = (x+p)^2.$$

Note $x^2 - 2qx + q^2 = (x-q)^2$ is also a perfect square.

Completing the square

In this section we'll learn about an ancient algebra technique called *completing the square*, which allows us to rewrite *any* quadratic expression of the form $x^2 + Bx + C$ as a perfect square plus some constant correction factor $(x+p)^2 + k$. This algebra technique was described in one of the first books on *al-jabr* (algebra), written by Al-Khwarizmi around the year 800 CE. The name "completing the square" comes from the ingenious geometric construction used by this procedure. Yes, we can use geometry to solve algebra problems!

We assume the starting point for the procedure is a quadratic expression whose quadratic coefficient is one, $1x^2 + Bx + C$, and use capital letters B and C to denote the linear and constant coefficients. The capital letters are to avoid any confusion with the quadratic expression $ax^2 + bx + c$, for which $a \neq 1$. Note we can always write $ax^2 + bx + c$ as $a(x^2 + \frac{b}{a}x + \frac{c}{a})$ and apply the procedure to the expression inside the brackets, identifying $\frac{b}{a}$ with B and $\frac{c}{a}$ with C.

First let's rewrite the quadratic expression $x^2 + Bx + C$ by splitting the linear term into two equal parts:

$$x^2 + \tfrac{B}{2}x + \tfrac{B}{2}x + C.$$

We can interpret the first three terms geometrically as follows: the x^2 term corresponds to a square with side length x, while the two $\frac{B}{2}x$ terms correspond to rectangles with sides $\frac{B}{2}$ and x. See the left side of Figure 1.15 for an illustration.

new illustration

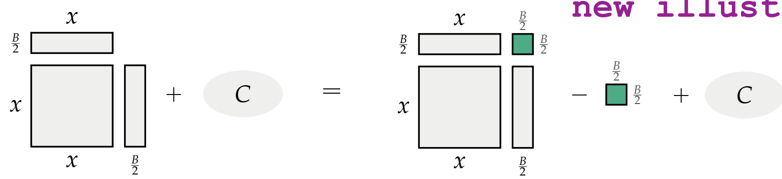

Figure 1.15: To complete the square in the expression $x^2 + Bx + C$, we need to add the quantity $(\frac{B}{2})^2$, which corresponds to a square (shown in darker colour) with sides equal to half the coefficient of the linear term. We also subtract $(\frac{B}{2})^2$ so the overall value of the expression remains unchanged.

The square with area x^2 and the two rectangles can be positioned to form a larger square with side length $\left(x + \frac{B}{2}\right)$. Note there's a small piece of sides $\frac{B}{2}$ by $\frac{B}{2}$ missing from the corner. To *complete the square*, we can add a term $\left(\frac{B}{2}\right)^2$ to this expression. To preserve the equality, we also subtract $\left(\frac{B}{2}\right)^2$ from the expression to obtain:

$$\begin{aligned} x^2 + \tfrac{B}{2}x + \tfrac{B}{2}x + C &= \underbrace{x^2 + \tfrac{B}{2}x + \tfrac{B}{2}x + \left(\tfrac{B}{2}\right)^2} - \left(\tfrac{B}{2}\right)^2 + C \\ &= \qquad \left(x + \tfrac{B}{2}\right)^2 \qquad - \left(\tfrac{B}{2}\right)^2 + C. \end{aligned}$$

The right-hand side of this equation describes the area of the square with side length $\left(x + \frac{B}{2}\right)$, minus the area of the small square $\left(\frac{B}{2}\right)^2$, plus the constant C, as illustrated on the right side of Figure 1.15.

We can summarize the entire procedure in one equation:

$$x^2 + Bx + C = \Big(x \underbrace{+ \tfrac{B}{2}}_{(1)}\Big)^2 + C \underbrace{- \left(\tfrac{B}{2}\right)^2}_{(2)}.$$

There are two things to remember when you want to apply the complete-the-square trick: (1) choose the constant inside the bracket to be $\frac{B}{2}$ (half of the linear coefficient), and (2) subtract $\left(\frac{B}{2}\right)^2$ outside the bracket in order to keep the equation balanced.

Solving quadratic equations

Suppose we want to solve the quadratic equation $x^2 + Bx + C = 0$. It's not possible to solve this equation with the digging-toward-the-x approach from Section 1.1 (since x appears in both the quadratic term x^2 and the linear term Bx). Enter the completing-the-square trick!

Example Let's find the solutions of the equation $x^2 + 5x + 6 = 0$. The coefficient of the linear term is $B = 5$, so we choose $\frac{B}{2} = \frac{5}{2}$ for the constant inside the bracket, and subtract $\left(\frac{B}{2}\right)^2 = \left(\frac{5}{2}\right)^2$ outside the bracket to keep the equation balanced. Completing the square gives

$$x^2 + 5x + 6 = \left(x + \tfrac{5}{2}\right)^2 + 6 - \left(\tfrac{5}{2}\right)^2 = 0.$$

Next we use fraction arithmetic to simplify the constant terms in the expression: $6 - \left(\frac{5}{2}\right)^2 = 6 \cdot \frac{4}{4} - \frac{25}{4} = \frac{24-25}{4} = \frac{-1}{4} = -0.25$.

We're left with the equation

$$(x + 2.5)^2 - 0.25 = 0,$$

which we can now solve by digging toward x. First move 0.25 to the right-hand side to get $(x + 2.5)^2 = 0.25$. Then take the square root on both sides to obtain $(x + 2.5) = \pm 0.5$, which simplifies to $x = -2.5 \pm 0.5$. The two solutions are $x = -2.5 + 0.5 = -2$ and $x = -2.5 - 0.5 = -3$. You can verify these solutions by substituting the values in the original equation $(-2)^2 + 5(-2) + 6 = 0$ and similarly $(-3)^2 + 5(-3) + 6 = 0$. Congratulations, you just solved a quadratic equation using a 1200-year-old algebra technique!

In the next section, we'll learn how to leverage the complete-the-square trick to obtain a general-purpose formula for quickly solving quadratic equations.