Geometric solution

Solving a system of two linear equations in two unknowns can be understood geometrically as finding the point of intersection between two lines in the Cartesian plane. In this section we'll explore this correspondence between algebra and geometry to develop yet another way of solving systems of linear equations.

The algebraic equation $ax + by = c$ containing the unknowns x and y can be interpreted as a *constraint* equation on the set of possible values for the variables x and y. We can visualize this constraint geometrically by considering the coordinate pairs (x, y) that lie in the Cartesian plane. Recall that every point in the Cartesian plane can be represented as a coordinate pair (x, y), where x and y are the coordinates of the point.

Figure 1.77 shows the geometrical representation of three equations. The line ℓ_a corresponds to the set of points (x, y) that satisfy the equation $x = 1$, the line ℓ_b is the set of points (x, y) that satisfy the equation $y = 2$, and the line ℓ_c corresponds to the set of points that satisfy $x + 2y = 2$.

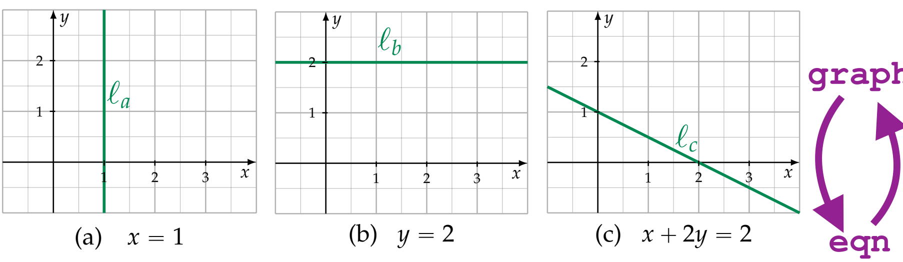

Figure 1.77: Graphical representations of three linear equations.

You can convince yourself that the geometric lines shown in Figure 1.77 are equivalent to the algebraic equations by considering individual points (x, y) in the plane. For example, the points $(1, 0)$, $(1, 1)$, and $(1, 2)$ are all part of the line ℓ_a since they satisfy the equation $x = 1$. For the line ℓ_c, you can verify that the line's x-intercept $(2, 0)$ and its y-intercept $(0, 1)$ both satisfy the equation $x + 2y = 2$.

The Cartesian plane as a whole corresponds to the set $\mathbb{R}^2$, which describes all possible pairs of coordinates. To understand the equivalence between the algebraic equation $ax + by = c$ and the line ℓ in the Cartesian plane, we can use the following precise math notation:

$$\ell : \{(x, y) \in \mathbb{R}^2 \mid ax + by = c\}.$$

In words, this means that the line ℓ is defined as the subset of the pairs of real numbers (x, y) that satisfy the equation $ax + by = c$.

Figure 1.78 shows the graphical representation of the line ℓ.

You don't have to take my word for it, though! Think about it and convince yourself that all points on the line ℓ shown in Figure 1.78 satisfy the equation $ax + by = c$. For example, you can check that the x-intercept $(\frac{c}{a}, 0)$ and the y-intercept $(0, \frac{c}{b})$ satisfy the equation $ax + by = c$.

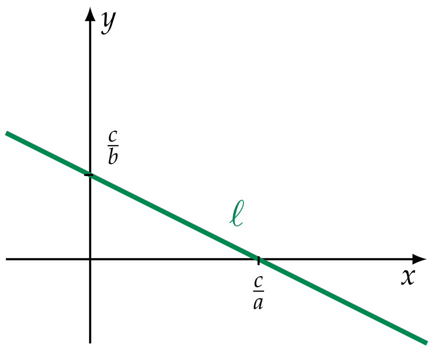

Figure 1.78: Graphical representation of the equation $ax + by = c$.

Solving the system of two equations

$$
\begin{aligned}
a_1 x + b_1 y &= c_1, \\
a_2 x + b_2 y &= c_2,
\end{aligned}
$$

corresponds to finding the intersection of the lines ℓ_1 and ℓ_2 that represent each equation. The pair (x, y) that satisfies both algebraic equations simultaneously is equivalent to the point (x, y) that is the intersection of lines ℓ_1 and ℓ_2, as illustrated in Figure 1.79.

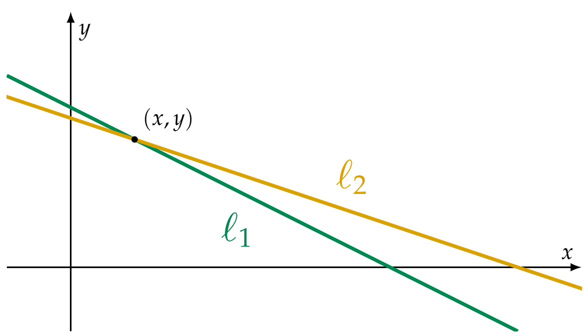

Figure 1.79: The point (x, y) that lies at the intersection of lines ℓ_1 and ℓ_2.

Example Let's see how we can use the geometric interpretation to solve the system of equations

$$\begin{aligned} x + 2y &= 5, \\ 3x + 9y &= 21. \end{aligned}$$

We've already seen three different *algebraic* techniques for finding the solution to this system of equations; now let's see a *geometric* approach for finding the solution. I'm not kidding you, we're going to solve the exact same system of equations a fourth time!

The first step is to draw the lines that correspond to each of the equations using pen and paper or a graphing calculator. The second step is to find the coordinates of the point where the two lines intersect as shown in Figure 1.80. The point $(1,2)$ that lies on both lines ℓ_1 and ℓ_2 corresponds to the x and y values that satisfy both equations simultaneously.

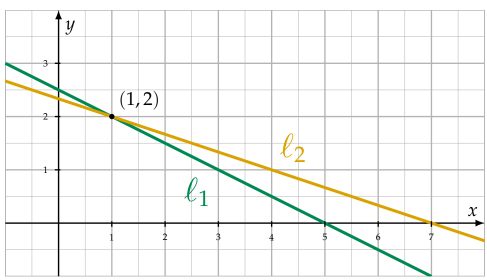

Figure 1.80: The line ℓ_1 with equations $x + 2y = 5$ intersects the line ℓ_2 with equation $3x + 9y = 21$ at the point $(1,2)$.

Visit the webpage at `www.desmos.com/calculator/exikik615f` to play with an interactive version of the graphs shown in Figure 1.80. Try changing the equations and see how the graphs change.

Exercises

E1.35 Plot the lines ℓ_a, ℓ_b, and ℓ_c shown in Figure 1.77 (page 132) using the Desmos graphing calculator. Use the graphical representation of these lines to find: **a)** the intersection of lines ℓ_c and ℓ_a, **b)** the intersection of ℓ_a and ℓ_b, and **c)** the intersection of lines ℓ_b and ℓ_c.

E1.36 Solve the system of equations simultaneously for x and y:

$$\begin{aligned} 2x + 4y &= 16, \\ 5x - y &= 7. \end{aligned}$$

1.22 Set notation

[FIVE PAGES MOSTLY NO CHANGES]

Sets as solutions to equations

Another context where sets come up is when describing solutions to equations and inequalities. In Section 1.1 we learned how to solve for the unknown x in equations. To solve the equation $f(x) = c$ is to find all the values of x that satisfy this equation. For simple equations like $x - 3 = 6$, the solution is a single number $x = 9$, but more complex equations can have multiple solutions. For example, the solution to the equation $x^2 = 4$ is the set $\{-2, 2\}$, since both $x = -2$ and $x = 2$ satisfy the equation.

Please update your definition of the math verb "to solve" (an equation) to include the new notion of a *solution set*—the set of values that satisfy the equation. A solution set is the mathematically precise way to describe an equation's solutions:

- The solution set to the equation $x - 3 = 6$ is the set $\{9\}$.
- The solution set for the equation $x^2 = 4$ is the set $\{-2, 2\}$.
- The solution set of $\sin(x) = 0$ is the set $\{x \mid x = \pi n, \forall n \in \mathbb{Z}\}$.
- The solution set for the equation $\sin(x) = 2$ is $\varnothing$ (the empty set), since there is no number x that satisfies the equation.

The SymPy function solve returns the solutions of equations as a list. To solve the equation $f(x) = c$ using SymPy, we first rewrite it as expression that equals zero $f(x) - c = 0$, then call the function solve:

```
>>> solve(x-3 -6, x)          # usage: solve(expr, var)
[9]

>>> solve(x**2 -4, x)
[-2, 2]

>>> solve(sin(x), x)
[0, pi]                        # found only solutions in [0,2*pi)

>>> solve(sin(x) -2, x)
[]                             # empty list = empty set
```

In the next section we'll learn how the notion of a solution set is used for describing the solutions to systems of equations.

Solution sets to systems of equations

Let's revisit what we learned in Section 1.21 about the solutions to systems of linear equations, and define their solution sets more precisely. The solution set for the system of equations

$$a_1x + b_1y = c_1,$$
$$a_2x + b_2y = c_2,$$

corresponds to the intersection of two sets:

$$\underbrace{\{(x,y) \in \mathbb{R}^2 \mid a_1x + b_1y = c_1\}}_{\ell_1} \cap \underbrace{\{(x,y) \in \mathbb{R}^2 \mid a_2x + b_2y = c_2\}}_{\ell_2}.$$

Recall that the lines ℓ_1 and ℓ_2 are the geometric interpretation of these sets. Each line corresponds to a set of coordinate pairs (x,y) that satisfy the equation of the line. The solution to the system of equations is the set of points at the intersection of the two lines $\ell_1 \cap \ell_2$. Note the word *intersection* is used in two different mathematical contexts: the solution is the *set intersection* of two sets, and also the *geometric intersection* of two lines.

Let's take advantage of this correspondence between set intersections and geometric line intersections to understand the solutions to systems of equations in a little more detail. In the next three sections, we'll look at three possible cases that can occur when trying to solve a system of two linear equations in two unknowns. So far we've only discussed Case A, which occurs when the two lines intersect at a point, as in the example shown in Figure 1.81. To fully understand the possible solutions to a system of equations, we need to think about all other cases; like Case B when $\ell_1 \cap \ell_2 = \varnothing$ as in Figure 1.82, and Case C when $\ell_1 \cap \ell_2 = \ell_1 = \ell_2$ as in Figure 1.83.

Case A: One solution. When the lines ℓ_1 and ℓ_2 are non-parallel, they will intersect at a point as shown in Figure 1.81. In this case, the solution set to the system of equations contains a single point:

$$\{(x,y) \in \mathbb{R}^2 \mid x + 2y = 2\} \cap \{(x,y) \in \mathbb{R}^2 \mid x = 1\} = \{(1, \tfrac{1}{2})\}.$$

Case B: No solution. If the lines ℓ_1 and ℓ_2 are parallel then they will never intersect. The intersection of these lines is the empty set:

$$\{(x,y) \in \mathbb{R}^2 \mid x + 2y = 2\} \cap \{(x,y) \in \mathbb{R}^2 \mid x + 2y = 4\} = \varnothing.$$

Think about it—there is no point (x,y) that lies on both ℓ_1 and ℓ_2. Using algebra terminology, we say this system of equations has no solution, since there are no numbers x and y that satisfy both equations.

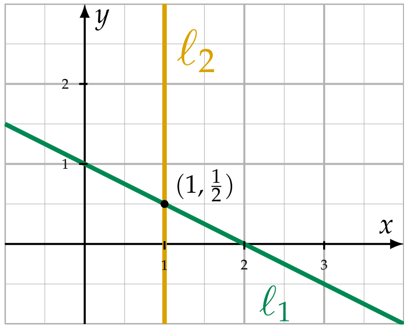

Figure 1.81: Case A: The intersection of the lines with equations $x + 2y = 2$ and $x = 1$ is the point $(1, \frac{1}{2}) \in \mathbb{R}^2$.

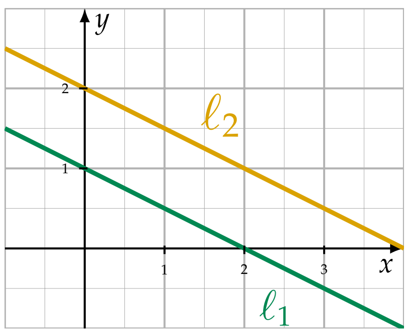

Figure 1.82: Case B: The lines with equations $x + 2y = 2$ and $x + 2y = 4$ are parallel and do not intersect. Using set notation, we can describe the solution set as $\varnothing$ (the empty set).

Case C: Infinitely many solutions. If the lines ℓ_1 and ℓ_2 are parallel and overlapping then they intersect everywhere. This case occurs when one of the equations in a system of equations is a multiple of the other equation, as in the case of equations $x + 2y = 2$ and $3x + 6y = 6$. The lines ℓ_1 and ℓ_2 that correspond to these equations are shown in Figure 1.83. Any point (x, y) that satisfies $x + 2y = 2$ also satisfies $3x + 6y = 6$. Since both equations describe the same geometric line, the intersection of the two lines is equal to the lines: $\ell_1 \cap \ell_2 = \ell_1 = \ell_2$. In this case, the solution to the system of equations is described by the set $\{(x, y) \in \mathbb{R}^2 \mid x + 2y = 2\}$.

We need to consider all three cases when thinking about the solutions to systems of linear equations: the solution set can be a point (Case A), the empty set (Case B), or a line (Case C). Observe that the same mathematical notion (a set) is able to describe the solutions in all three cases even though the solutions correspond to very different geometric objects. In Case A the solution is a set that con-

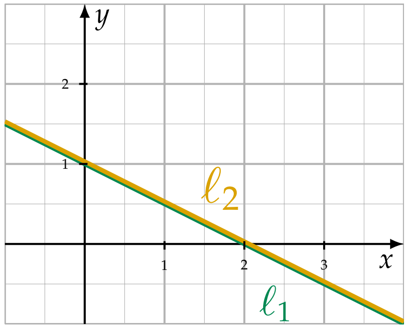

Figure 1.83: Case C: the line ℓ_1 described by equation $x + 2y = 2$ and the line ℓ_2 described by equation $3x + 6y = 6$ correspond to the same line in the Cartesian plane. The intersection of these lines is the set $\{(x, y) \in \mathbb{R}^2 \mid x + 2y = 2\} = \ell_1 = \ell_2$.

tains a single point $\{(x, y)\}$. In Case B the solution is the empty set $\emptyset$. And in Case C the solution set is described by the infinite set $\{(x, y) \in \mathbb{R}^2 \mid ax + by = c\}$, which corresponds to a line ℓ in the Cartesian plane. I hope you'll agree with me that set notation is useful for describing mathematical concepts precisely and for handling solutions to linear equations.