

Discussion

We did a lot of hands-on activities with vectors in this section and skipped over some of the theoretical details. Now that you've been exposed to the practical side of vector calculations, it's worth clarifying certain points that we glossed over.

Vectors vs. points

We used the notation \mathbb{R}^2 to describe two kinds of math objects: the set of points in the Cartesian plane and the set of vectors in a two-dimensional space. The point $P = (P_x, P_y)$ and the vector $\vec{v} = (v_x, v_y)$ are both represented by pairs of real numbers, so we use the notation $P \in \mathbb{R}^2$ and $\vec{v} \in \mathbb{R}^2$ to describe them. This means that a pair of numbers $(3, 2) \in \mathbb{R}^2$ could represent the *coordinates* of a point, or the *components* of a vector, depending on the context.

Let's take a moment to review the definitions of points and vectors and clarify the types of operations we can perform on them:

- **Space of points** \mathbb{R}^2 : the set of points $P = (P_x, P_y)$ corresponds to locations in the Cartesian plane. The point $P = (P_x, P_y)$ corresponds to the geometric instructions: "Starting at the origin $(0, 0)$, move P_x units along the x -axis and P_y units along the y -axis." The distance between points P and Q is denoted $d(P, Q)$.
- **Vector space** \mathbb{R}^2 : the set of vectors $\vec{v} = (v_x, v_y)$ describes displacements in the Cartesian plane. The vector $\vec{v} = (v_x, v_y)$ corresponds to the instructions: "Starting anywhere, move v_x units along the x -axis and v_y units along the y -axis." Vectors can be combined and manipulated using the vector algebra operations $\vec{u} + \vec{v}$, $\vec{u} - \vec{v}$, $\alpha\vec{u}$, $\vec{u} \cdot \vec{v}$, and $\|\vec{v}\|$.

Note the geometric instructions for points and vectors are very similar; the only difference is the starting point. The coordinates of a point (P_x, P_y) specify a *fixed position* relative to the origin $(0, 0)$, while the components of a vector (v_x, v_y) describe a *relative displacement* that can have any starting point.

Let's look at some examples of calculations that combine points and vectors. Consider the points P and Q in the Cartesian plane, and the displacement vector \vec{v}_{PQ} between them. The displacement vector \vec{v}_{PQ} gives the "move instructions" for getting from point P to point Q and is defined by the equation:

$$\vec{v}_{PQ} = Q - P.$$

This equation says that subtracting two points produces a vector, which make sense if you think about it—the "difference" between two points is a displacement vector.

We can use the displacement vector \vec{v}_{PQ} in calculations like this:

$$P + \vec{v}_{PQ} = P + (Q - P) = Q.$$

In words, this calculation shows that “Starting at the point P and moving by \vec{v}_{PQ} brings us to the point Q .”

The above equations use addition and subtraction operations between a mix of points and vectors. This is rather unusual: normally we only use operations like “+” and “-” between math objects of the same kind. In this case, we’re allowed to mix points and vectors because they both describe “move instructions” of the same kind.

Let’s keep going. What other useful calculations can we do by combining points and vectors? Suppose we wanted to find the midpoint M that lies exactly in the middle between points P and Q . We can find the midpoint M using the displacement vector \vec{v}_{PQ} and some basic vector algebra. If starting from P and moving by \vec{v}_{PQ} brings us all the way to the point Q , then starting from P and moving by $\frac{1}{2}\vec{v}_{PQ}$ will bring us to the midpoint: $M = P + \frac{1}{2}\vec{v}_{PQ}$.

The mathematical bridge between points and vectors allows us to use vector techniques to solve geometry problems. By learning to describe geometric objects like points, lines, and circles using vectors, we can do complicated geometry calculations using simple algebraic manipulations like vector operations. This exemplifies a general pattern in mathematics: applying techniques developed in one domain to solve problems in another domain.

Example You come to class one day and there’s a surprise quiz that asks you to write the formula for the distance $d(P, Q)$ between two points $P = (P_x, P_y)$ and $Q = (Q_x, Q_y)$. You don’t remember ever learning about such a formula and feel caught off guard. How can the teacher ask for a formula they haven’t covered in class yet? This seems totally unfair!

After a minute of stressing out, you take a deep breath, come back to your senses, and resolve to give this problem a shot. You start by sketching a coordinate system, placing points P and Q in it, and drawing the line that connects the two points. What is the formula that describes the length of this line?

The line from P to Q looks like the hypotenuse of a triangle, which makes you think that trigonometry could somehow be used to find the answer. Unfortunately, trying to remember the trigonometry formulas has only the effect of increasing your math anxiety. You take this as a sign that you should look for other options. In math, it’s important to trust your gut instincts.

By a fortunate coincidence, you were recently reading about the connection between points and vectors, and specifically about the

displacement vector $\vec{v}_{PQ} = Q - P$. The line in your sketch represents the vector \vec{v}_{PQ} . You realize that the distance between the points P and Q is the same as the length of the vector \vec{v}_{PQ} . You remember the formula for the length of a vector \vec{v} is $\|\vec{v}\| = \sqrt{v_x^2 + v_y^2}$ and you know the formula for the displacement vector is $\vec{v}_{PQ} = (Q_x - P_x, Q_y - P_y)$, so you combine these formulas to obtain the answer: $d(P, Q) = \|\vec{v}_{PQ}\| = \sqrt{(Q_x - P_x)^2 + (Q_y - P_y)^2}$. One more win for the “don’t worry and try it” strategy for solving math problems!

Vectors and vector coordinates

One final point we need to clarify is the difference between real-world vector quantities like the velocity of a tennis ball \vec{v} and its mathematical representation as a coordinate vector (v_x, v_y, v_z) . If you know the coordinate vector (v_x, v_y, v_z) then you know what the real-world velocity is, right? Not quite.

Let's say you're doing a physics research project on tennis serves. You define an xyz -coordinate system for the tennis court, which allows you to represent the ball's velocity \vec{v} as a triple of components (v_x, v_y, v_z) interpreted as: "The ball is moving with velocity v_x units in the x -direction, v_y units in the y -direction, and v_z units in the z -direction."

Suppose you want to describe the velocity vector \vec{v} to a fellow physicist via text message. Referring to your sheet of calculations, you find the values $\vec{v} = (60, 3, -2)$, which you know were measured in metres per second. You send this message:

The velocity is $(60, 3, -2)$ measured in metres per second.

A few minutes later the following reply comes back:

Wait whaaat? What coordinate system are you using?

Indeed the information you sent is incomplete. Vector components depend on the coordinate system in which the vectors are represented. The triple of numbers $(60, 3, -2)$ only makes sense once you know the directions of the axes in the xyz -coordinate system. Realizing your mistake, you send a text with all the required information:

Using the coordinate system centred at the south post of the net, with the x -axis pointing east along the court, the y -axis pointing north along the net, and the z -axis pointing up, the velocity is $(60, 3, -2)$ in metres per second.

A few seconds later, you get the reply:

OK got it now. Thx!

This hypothetical situation illustrates the importance of the coordinate systems for describing vectors. If you don't know what the coordinate system is, knowing the coordinates (v_x, v_y, v_z) doesn't tell you much. Only when you know the directions of the unit vectors \hat{i} , \hat{j} , and \hat{k} can you interpret the instructions $\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$.

It turns out, using the xyz -coordinate system with the three vectors $\{\hat{i}, \hat{j}, \hat{k}\}$ is just one of many possible ways we can represent vectors. We can represent a vector \vec{v} as coordinates (v_1, v_2, v_3) with respect to any *basis* $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ using the expression $\vec{v} = v_1\hat{e}_1 + v_2\hat{e}_2 + v_3\hat{e}_3$, which corresponds to the instructions: "Move v_1 units in the direction of \hat{e}_1 , move v_2 units in the direction of \hat{e}_2 , and move v_3 units in the direction of \hat{e}_3 ."

Discussion

It would be hard to over-emphasize the importance of the basis—the coordinate system you use to describe vectors. The choice of coordinate system is the bridge between real-world vector quantities and their mathematical representation in terms of components. Every time you start a new problem that involves vector calculations, the first thing you should do is choose the coordinate system you want to use, and indicate it clearly in the diagram.

Using a non-standard coordinate system can sometimes simplify the equations you have to solve. For example, let's say we want to study the motion of a block sliding down an incline with velocity \vec{v} , as illustrated in Figure 3.11. Using the standard xy -basis, the velocity vector is represented as $(v \cos \theta, -v \sin \theta)_{xy}$, which has components in both the x - and y -directions and requires using trigonometric functions. If instead you use the non-standard $x'y'$ -basis, the components of the velocity will be $(v, 0)_{x'y'}$. Note the velocity only has a component along the x' -direction, which will simplify all subsequent calculations.

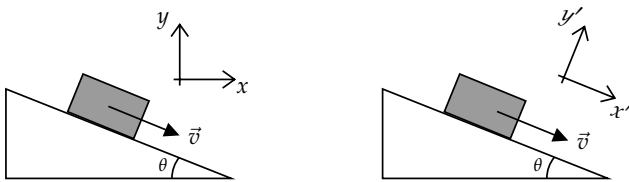


Figure 3.11: The vector \vec{v} is described by the coordinates $(v \cos \theta, -v \sin \theta)_{xy}$ with respect to the standard basis xy . The same vector \vec{v} is described by the coordinates $(v, 0)_{x'y'}$ with respect to the “tilted” basis $x'y'$.

Recall the polar coordinates representation we used to describe points $r\angle\theta$ and vectors $\|\vec{v}\|\angle\theta$ in two dimensions (see page 187). This is another example of an alternative coordinate system that's useful for describing rotations and circular motion. Note certain textbooks will write the polar coordinates of the vector $\vec{v} = \|\vec{v}\|\angle\theta$ using the bracket notation $(\|\vec{v}\|, \theta)$, which can easily be confused with the Cartesian coordinates of the vector (v_x, v_y) . Indicating the coordinate system as a subscript after the bracket can avoid any confusion: $\vec{v} = (\|\vec{v}\|, \theta)_{r\theta} = (v_x, v_y)_{xy}$.