Chapter 5

Functions

Learning math is like learning a new language. One of the most useful ideas you can learn in the language of math is the concept of a function. If numbers and variables correspond to nouns, functions correspond to verbs. Specifically, functions codify relations between variables; any relationship between two variables can be expressed as a function. This makes functions a powerful tool for modelling real-world situations.

In this chapter we'll learn some important math verbs. We'll start with some general theory about functions, then catalogue the most important functions that occur in everyday life, in scientific modelling, and in business. We're building your vocabulary of math action verbs so you'll be able to get in on the action.

In Section 5.1 we'll introduce the main concepts used to characterize functions: the function's domain and image, the function's graph, and the function's values. Section 5.2 serves as a reference manual for 10 functions of central importance in math, science, and engineering. In Section 5.3 we'll observe how translation and scaling transformations affect the functions' graphs.

5.1 Functions

We need to have a relationship talk. We need to talk about functions. We use functions to describe the relationships between variables. In particular, functions describe how one variable *depends* on another.

For example, the revenue R from a music concert depends on the number of tickets sold n. If each ticket costs \$25, the revenue from the concert can be written *as a function of* n as follows: $R(n) = 25n$. Solving for n in the equation $R(n) = 7000$ tells us the number of ticket sales needed to generate \$7000 in revenue. This is a simple model of a function; as your knowledge of functions builds, you'll learn how to build more detailed models of reality. For instance, if you need to include a 5% processing charge for issuing the tickets, you can update the revenue model to $R(n) = 0.95 \cdot 25 \cdot n$. If the estimated cost of hosting the concert is $C = \$2000$, then the profit from the concert P can be modelled as

$$\begin{aligned} P(n) &= R(n) \; - \; C \\ &= 0.95 \cdot \$25 \cdot n \; - \; \$2000 \end{aligned}$$

The function $P(n) = 23.75n - 2000$ models the profit from the concert as a function of the number of tickets sold. This is a pretty good model already, and you can always update it later as you learn more information.

The more functions you know, the more tools you have for modelling reality. To "know" a function, you must be able to understand and connect several of its aspects. First you need to know the function's mathematical **definition**, which describes exactly what the function does. Starting from the function's definition, you can use your existing math skills to find the function's **properties**. You must also know the **graph** of the function; what the function looks like if you plot x versus $f(x)$ in the Cartesian plane. It's also a good idea to remember the **values** of the function for some important inputs. Finally—and this is the part that takes time—you must learn about the function's **relations** to other functions.

Definitions

A *function* is a mathematical object that takes numbers as inputs and produces numbers as outputs. We use the notation

$$f\colon A \to B$$

to denote a function from the input set A to the output set B. In this book, we mostly study functions that take real numbers as inputs and give real numbers as outputs: $f\colon \mathbb{R} \to \mathbb{R}$.
A function is not a number; rather, it is a *mapping* from numbers to numbers. We say "f maps x to $f(x)$." For any input x, the output value of f for that input is denoted $f(x)$, which is read as "f of x."

We'll now define some fancy technical terms used to describe the input and output sets of functions.

- A: the *source set* of the function describes the types of numbers that the function takes as inputs.
- $\text{Dom}(f)$: the *domain* of a function is the set of allowed input values for the function.

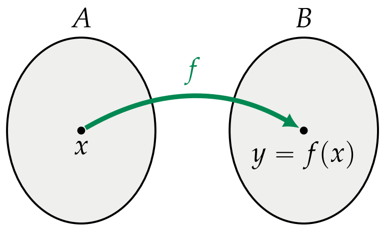


Figure 5.1: An abstract representation of a function f from the set A to the set B. The function f is the arrow which *maps* each input x in A to an output $f(x)$ in B. The output of the function $f(x)$ is also denoted y.

- B: the *target set* of a function describes the type of outputs the function has. The target set is sometimes called the *codomain*.
- $\text{Im}(f)$: the *image* of the function is the set of all possible output values of the function. The image is sometimes called the *range*.

See Figure 5.2 for an illustration of these concepts. The purpose of introducing all this math terminology is so we'll have words to distinguish the general types of inputs and outputs of the function (real numbers, complex numbers, vectors) from the specific properties of the function like its domain and image.

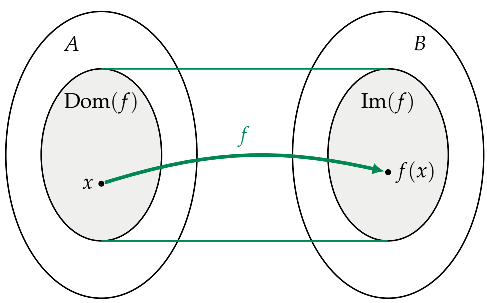


Figure 5.2: Illustration of the input and output sets of a function $f: A \to B$. The *source set* is denoted A and the *domain* is denoted $\text{Dom}(f)$. Note that the function's domain is a subset of its source set. The *target set* is denoted B and the *image* is denoted $\text{Im}(f)$. The image is a subset of the target set.

Let's look at an example to illustrate the difference between the source set and the domain of a function. Consider the square root function $f: \mathbb{R} \to \mathbb{R}$ defined as $f(x) = \sqrt{x}$, which is shown in Figure 5.3. The source set of f is the set of real numbers—yet only nonnegative real numbers are allowed as inputs, since $\sqrt{x}$ is not defined for negative numbers. Therefore, the domain of the square root function is only the nonnegative real numbers: $\text{Dom}(f) = \mathbb{R}_+ = \{x \in$

$\mathbb{R} \mid x \geqslant 0\}$. Knowing the domain of a function is essential to using the function correctly. In this case, whenever you use the square root function, you need to make sure that the inputs to the function are nonnegative numbers.

The complicated-looking expression between the curly brackets uses *set notation* to define the set of nonnegative numbers $\mathbb{R}_+$. In words, the expression $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geqslant 0\}$ states that "$\mathbb{R}_+$ is defined as the set of all real numbers x such that x is greater than or equal to zero." We'll discuss set notation in more detail in Section 8.3. For now, you can just remember that $\mathbb{R}_+$ represents the set of nonnegative real numbers.

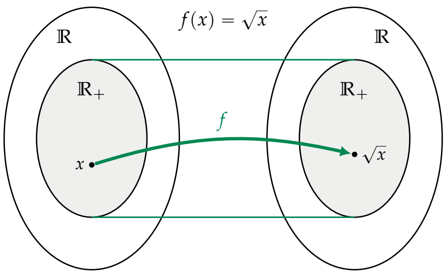


Figure 5.3: The input and output sets of the function $f(x) = \sqrt{x}$. The domain of f is the set of nonnegative real numbers $\mathbb{R}_+$ and its image is $\mathbb{R}_+$.

To illustrate the difference between the image of a function and its target set, let's look at the function $f(x) = x^2$ shown in Figure 5.4. The quadratic function is of the form $f\colon \mathbb{R} \to \mathbb{R}$. The function's source set is $\mathbb{R}$ (it takes real numbers as inputs) and its target set is $\mathbb{R}$ (the outputs are real numbers too); however, not all real numbers are possible outputs. The *image* of the function $f(x) = x^2$ consists only of the nonnegative real numbers $\mathbb{R}_+ = \{y \in \mathbb{R} \mid y \geqslant 0\}$, since $f(x) \geqslant 0$ for all x.

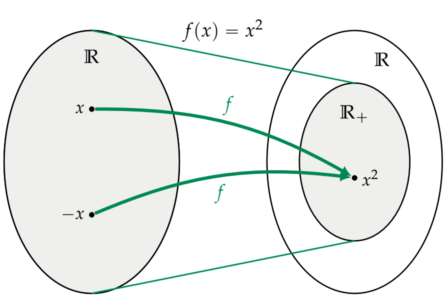


Figure 5.4: The function $f(x) = x^2$ is defined for all reals: $\text{Dom}(f) = \mathbb{R}$. The image of the function is the set of nonnegative real numbers: $\text{Im}(f) = \mathbb{R}_+$.

Function properties

We'll now introduce some additional terminology for describing three important function properties. Every function is a mapping from a source set to a target set, but what kind of mapping is it?

- A function is *injective* if it maps two different inputs to two different outputs. If x_1 and x_2 are two input values that are not equal $x_1 \neq x_2$, then the output values of an injective function will also not be equal $f(x_1) \neq f(x_2)$.
- A function is *surjective* if its image is equal to its target set. For every output y in the target set of a surjective function, there is at least one input x in its domain such that $f(x) = y$.
- A function is *bijective* if it is both injective and surjective.

I know this seems like a lot of terminology to get acquainted with, but it's important to have names for these function properties. We'll need this terminology to give a precise definition of the *inverse function* in the next section.

Injective property We can think of *injective* functions as pipes that transport fluids between containers. Since fluids cannot be compressed, the "output container" must be at least as large as the "input container." If there are two distinct points x_1 and x_2 in the input container of an injective function, then there will be two distinct points $f(x_1)$ and $f(x_2)$ in the output container of the function as well. In other words, injective functions don't smoosh things together.

In contrast, a function that doesn't have the injective property can map several different inputs to the same output value. The function $f(x) = x^2$ is not injective since it sends inputs x and $-x$ to the same output value $f(x) = f(-x) = x^2$, as illustrated in Figure 5.4.

The maps-distinct-inputs-to-distinct-outputs property of injective functions has an important consequence: given the output of an injective function y, there is only one input x such that $f(x) = y$. If a second input x' existed that also leads to the same output $f(x) = f(x') = y$, then the function f wouldn't be injective. For each of the outputs y of an injective function f, there is a *unique* input x such that $f(x) = y$. In other words, injective functions have a unique-input-for-each-output property.

Surjective property A function is *surjective* if its outputs cover the entire target set: every number in the target set is a possible output of the function for some input. For example, the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ is surjective: for every number y in the target set $\mathbb{R}$, there is an input x, namely $x = \sqrt[3]{y}$, such that $f(x) = y$. The function $f(x) = x^3$ is surjective since its image is equal to its target set, $\text{Im}(f) = \mathbb{R}$, as shown in Figure 5.5.

On the other hand, the function $f: \mathbb{R} \to \mathbb{R}$ defined by the equation $f(x) = x^2$ is not surjective since its image is only the nonnegative numbers $\mathbb{R}_+$ and not the whole set of real numbers (see Figure 5.4). The outputs of this function do not include the negative numbers of the target set, because there is no real number x that can be used as an input to obtain a negative output value.

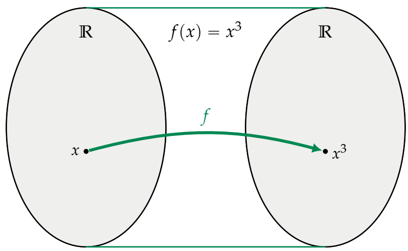


Figure 5.5: For the function $f(x) = x^3$ the image is equal to the target set of the function, $\text{Im}(f) = \mathbb{R}$, therefore the function f is surjective. The function f maps two different inputs $x_1 \neq x_2$ to two different outputs $f(x_1) \neq f(x_2)$, so f is injective. Since f is both injective and surjective, it is a *bijective* function.

Bijective property A function is bijective if it is both injective and surjective. When a function $f : A \to B$ has both the injective and surjective properties, it defines a *one-to-one correspondence* between the numbers of the source set A and the numbers of the target set B. This means for every input value x, there is exactly one corresponding output value y, and for every output value y, there is exactly one input value x such that $f(x) = y$. An example of a bijective function is the function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ (see Figure 5.5). For every input x in the source set $\mathbb{R}$, the corresponding output y is given by $y = f(x) = x^3$. For every output value y in the target set $\mathbb{R}$, the corresponding input value x is given by $x = \sqrt[3]{y}$.

A function is not bijective if it lacks one of the required properties. Examples of non-bijective functions are $f(x) = \sqrt{x}$, which is not surjective and $f(x) = x^2$, which is neither injective nor surjective.

Counting solutions Another way to understand the injective, surjective, and bijective properties of functions is to think about the solutions to the equation $f(x) = b$, where b is a number in the target set B. The function f is injective if the equation $f(x) = b$ has *at most one* solution for every number b. The function f is surjective if the equation $f(x) = b$ has *at least one* solution for every number b. If the function f is bijective then it is both injective and surjective, which means the equation $f(x) = b$ has *exactly one* solution.

Inverse function

We used inverse functions repeatedly in previous chapters, each time describing the inverse function informally as an "undo" operation. Now that we have learned about bijective functions, we can give a the precise definition of the inverse function and explain some of the details we glossed over previously.

Recall that a *bijective* function $f : A \to B$ is a *one-to-one correspondence* between the numbers in the source set A and numbers in the target set B: for every output y, there is exactly one corresponding input value x such that $f(x) = y$. The *inverse function*, denoted f^{-1}, is the function that takes any output value y in the set B and finds the corresponding input value x that produced it $f^{-1}(y) = x$.

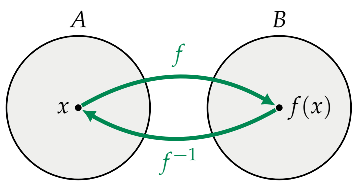


Figure 5.6: The inverse f^{-1} undoes the operation of the function f.

For every bijective function $f : A \to B$, there exists an inverse function $f^{-1} : B \to A$ that performs the *inverse mapping* of f. If we start from some x, apply f, and then apply f^{-1}, we'll arrive—full circle—back to the original input x:

$$f^{-1}\big(f(x) \big) = x.$$

In Figure 5.6 the function f is represented as a forward arrow, and the inverse function f^{-1} is represented as a backward arrow that puts the value $f(x)$ back to the x it came from.

Similarly, we can start from any y in the set B and apply f^{-1} followed by f to get back to the original y we started from:

$$f\big(f^{-1}(y) \big) = y.$$

In words, this equation tells us that f is the "undo" operation for the function f^{-1}, the same way f^{-1} is the "undo" operation for f.

If a function is missing the injective property or the surjective property then it isn't bijective and it doesn't have an inverse. Without the injective property, there could be two inputs x and x' that both produce the same output $f(x) = f(x') = y$. In this case, computing $f^{-1}(y)$ would be impossible since we don't know which of the two possible inputs x or x' was used to produce the output y. Without the surjective property, there could be some output y' in B for which the inverse function f^{-1} is not defined, so the equation $f(f^{-1}(y)) = y$ would not hold for all y in B. The inverse function f^{-1} exists only when the function f is bijective.

Wait a minute! We know the function $f(x) = x^2$ is not bijective and therefore doesn't have an inverse, but we've repeatedly used the square root function as an inverse function for $f(x) = x^2$. What's going on here? Are we using a double standard like a politician that espouses one set of rules publicly, but follows a different set of rules in their private dealings? Is mathematics corrupt?

Don't worry, mathematics is not corrupt—it's all legit. We can use inverses for non-bijective functions by imposing *restrictions* on the source and target sets. The function $f(x) = x^2$ is not bijective when defined as a function $f : \mathbb{R} \to \mathbb{R}$, but it *is* bijective if we define it as a function from the set of nonnegative numbers to the set of nonnegative numbers, $f : \mathbb{R}_+ \to \mathbb{R}_+$. Restricting the source set to $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geqslant 0\}$ makes the function injective, and restricting the target set to $\mathbb{R}_+$ also makes the function surjective. The function $f : \mathbb{R}_+ \to \mathbb{R}_+$ defined by the equation $f(x) = x^2$ is bijective and its inverse is $f^{-1}(y) = \sqrt{y}$.

It's important to keep track the of restrictions on the source set we applied when solving equations. For example, solving the equation $x^2 = c$ by restricting the solution space to nonnegative numbers will give us only the positive solution $x = \sqrt{c}$. We have to manually add the negative solution $x = -\sqrt{c}$ in order to obtain the complete solutions: $x = \sqrt{c}$ or $x = -\sqrt{c}$, which is usually written $x = \pm\sqrt{c}$. The possibility of multiple solutions is present whenever we solve equations involving non-injective functions.

Function composition

We can combine two simple functions by chaining them together to build a more complicated function. This act of applying one function after another is called *function composition*. Consider for example the composition:

$$f \circ g\,(x) = f(\,g(x)\,) = z.$$

Figure 5.7 illustrates the concept of function composition. First, the function $g : A \to B$ acts on some input x to produce an intermediary value $y = g(x)$ in the set B. The intermediary value y is then passed

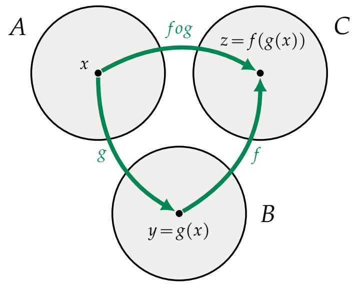


Figure 5.7: The function composition $f \circ g$ describes the combination of first applying the function g, followed by the function f: $f \circ g\,(x) = f(g(x))$.

through the function $f : B \to C$ to produce the final output value $z = f(y) = f(g(x))$ in the set C. We can think of the *composite function* $f \circ g$ as a function in its own right. The function $f \circ g : A \to C$ is defined through the formula $f \circ g\,(x) = f(g(x))$.

Don't worry too much about the "$\circ$" symbol—it's just a convenient math notation I wanted you to know about. Writing $f \circ g$ is the same as writing $f(g(x))$. The important takeaway from Figure 5.7 is that functions can be combined by using the outputs of one function as the inputs to the next. This is a very useful idea for building math models. You can understand many complicated input-output transformations by describing them as compositions of simple functions.

Example 1 Consider the function $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ given by $g(x) = \sqrt{x}$, and the function $f\colon \mathbb{R} \to \mathbb{R}_+$ defined by $f(x) = x^2$. The composite function $f \circ g\,(x) = (\sqrt{x})^2 = x$ is defined for all nonnegative reals. The composite function $g \circ f$ is defined for all real numbers, and we have $g \circ f\,(x) = \sqrt{x^2} = |x|$.

Example 2 The composite functions $f \circ g$ and $g \circ f$ describe different operations. If $g(x) = \ln(x)$ and $f(x) = x^2$, the functions $g \circ f\,(x) = \ln(x^2)$ and $f \circ g\,(x) = (\ln x)^2$ have different domains and produce different outputs, as you can verify using a calculator.

Using the notation "$\circ$" for function composition, we can give a concise description of the properties of a bijective function $f : A \to B$ and its inverse function $f^{-1} : B \to A$:

$$(f^{-1} \circ f)(x) = x \qquad \text{and} \qquad (f \circ f^{-1})(y) = y,$$

for all x in A and all y in B.

Function names

We use short symbols like $+$, $-$, $\times$, and $\div$ to denote most of the important functions used in everyday life. We also use the squiggle notation $\sqrt{\ }$ for square roots and superscripts to denote exponents. All other functions are identified and denoted by their *name*. If I want to compute the *cosine* of the angle $60°$ (a function describing the ratio between the length of one side of a right-angle triangle and the hypotenuse), I write $\cos(60°)$, which means I want the value of the cos function for the input $60°$.

Incidentally, the function cos has a nice output value for that specific angle: $\cos(60°) = \frac{1}{2}$. Therefore, seeing $\cos(60°)$ somewhere in an equation is the same as seeing $\frac{1}{2}$. To find other values of the function, say $\cos(33.13°)$, you'll need a calculator. All scientific calculators have a convenient little [cos] button for this very purpose.

Handles on functions

When you learn about functions you learn about the different "handles" by which you can "grab" these mathematical objects. The main handle for a function is its **definition**: it tells you the precise way to calculate the output when you know the input. The function definition is an important handle, but it is also important to "feel" what the function does intuitively. How does one get a feel for a function?

Table of values

One simple way to represent a function is to look at a list of input-output pairs: $\{\{\text{in} = x_1, \text{out} = f(x_1)\}, \{\text{in} = x_2, \text{out} = f(x_2)\}, \{\text{in} = x_3, \text{out} = f(x_3)\}, \ldots\}$. A more compact notation for the input-output pairs is $\{(x_1, f(x_1)), (x_2, f(x_2)), (x_3, f(x_3)), \ldots\}$, where the first number of each pair represents an input value and the second represents the output value given by the function.

We can also build a **table of values** by writing the input values in one column and recording the corresponding output values in a second column. You can choose inputs at random or focus on the important-looking x values in the function's domain.

You can create a table of values for any function you want to study. Follow the example shown in Table 5.1. Use the input values that interest you and fill out the right side of the table by calculating the value of $f(x)$ for each input x.

input $= x$	$\to$	$f(x) =$ output
0	$\to$	$f(0)$
1	$\to$	$f(1)$
55	$\to$	$f(55)$
x_4	$\to$	$f(x_4)$

Table 5.1: Table of input-output values of the function $f(x)$. The input values $x = 0$, $x = 1$ and $x = 55$ are chosen to "test" what the function does.

Function graph

One of the best ways to feel a function is to look at its graph. A graph is a line on a piece of paper that passes through all input-output pairs of a function. Imagine you have a piece of paper, and on it you draw a blank *coordinate system* as in Figure 5.8.

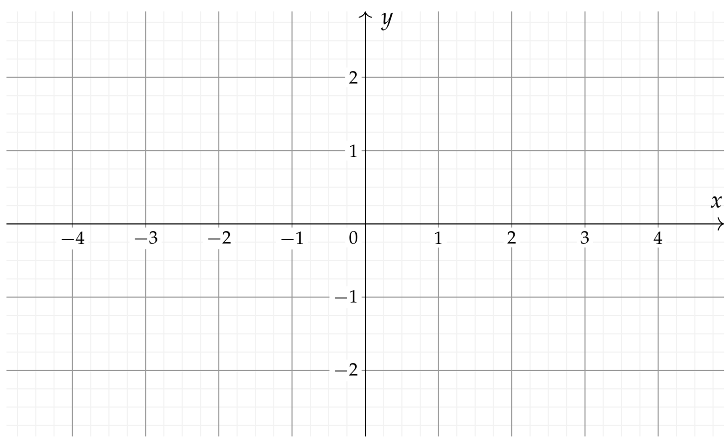


Figure 5.8: An empty (x, y)-coordinate system that you can use to draw function graphs. The graph of $f(x)$ consists of all the points for which $(x, y) = (x, f(x))$. See Figure 4.4 on page 56 for the graph of $f(x) = x^2$.

The horizontal axis is used to measure x. The vertical axis is used to measure $f(x)$. Because writing out $f(x)$ every time is long and tedious, we use a short, single-letter alias to denote the output value of f as follows:

$$y = f(x) = \text{output}.$$

Think of each input-output pair of the function f as a point (x, y) in the coordinate system. The graph of a function is a representational drawing of everything the function does. If you understand how to interpret this drawing, you can infer everything there is to know about the function.

Facts and properties

Another way to feel a function is by knowing the function's properties. This approach boils down to learning facts about the function and its connections to other functions. An example of a mathematical connection is the equation $\log_B(x) = \frac{\log_b(x)}{\log_b(B)}$, which describes a link between the logarithmic function base B and the logarithmic function base b.

The more you know about a function, the more "paths" your brain builds to connect to that function. Real math knowledge is not about memorization; it is about establishing a network of associations between different areas of information in your brain. See the concept maps on page iii for an illustration of the paths that link math concepts. Mathematical thought is the usage of these associations to carry out calculations and produce mathematical arguments. For example, knowing about the connection between logarithmic functions will allow you compute the value of $\log_7(e^3)$, even though calculators don't have a button for logarithms base 7. We find $\log_7(e^3) = \frac{\ln e^3}{\ln 7} = \frac{3}{\ln 7}$, which can be computed using the $\boxed{\ln}$ button.

To develop mathematical skills, it is vital to practice path-building between concepts by solving exercises. With this book, I will introduce you to some of the many paths linking math concepts, but it's on you to reinforce these paths through practice.

Example 3 Consider the function f from the real numbers to the real numbers ($f\colon \mathbb{R} \to \mathbb{R}$) defined as $f(x) = x^2 + 2x - 3$. The value of f when $x = 1$ is $f(1) = 1^2 + 2(1) - 3 = 0$. When $x = 2$, the output is $f(2) = 2^2 + 2(2) - 3 = 5$. What is the value of f when $x = 0$? You can use algebra to rewrite this function as $f(x) = (x+3)(x-1)$, which tells you the graph of this function crosses the x-axis at $x = -3$ and at $x = 1$. The values above will help you plot the graph of $f(x)$.

Example 4 Consider the exponential function with base 2 defined by $f(x) = 2^x$. This function is crucial to computer systems. For instance, RAM memory chips come in powers of two because the memory space is exponential in the number of "address lines" used on the chip. When $x = 1$, $f(1) = 2^1 = 2$. When x is 2 we have $f(2) = 2^2 = 4$. The function is therefore described by the following input-output pairs: $(0,1)$, $(1,2)$, $(2,4)$, $(3,8)$, $(4,16)$, $(5,32)$, $(6,64)$, $(7,128)$, $(8,256)$, $(9,512)$, $(10,1024)$, $(11,2048)$, $(12,4096)$, etc. Recall that any number raised to exponent 0 gives 1. Thus, the exponential function passes through the point $(0,1)$. Recall also that negative exponents lead to fractions, so we have the points $(-1,\frac{1}{2})$, $(-2,\frac{1}{4})$, $(-3,\frac{1}{8})$, etc. You can plot these $(x, f(x))$ coordinates in the Cartesian plane to obtain the graph of the function.

Discussion

To describe a function we specify its source and target sets $f\colon A \to B$, then give an equation of the form $f(x) =$ "expression involving x" that defines the function. Since functions are defined using equations, does this mean that functions and equations are the same thing? Let's take a closer look.

In general, any equation containing two variables describes a *relation* between these variables. For example, the equation $x - 3 = y - 4$ describes a relation between the variables x and y. We can isolate the variable y in this equation to obtain $y = x + 1$ and thus find the value of y when the value of x is given. We can also isolate x to obtain $x = y - 1$ and use this equation to find x when the value of y is given. In the context of an equation, the relationship between the variables x and y is symmetrical and no special significance is attached to either of the two variables.

We also can describe the same relationship between x and y as a function $f : \mathbb{R} \to \mathbb{R}$. We choose to identify x as the input variable and y as the output variable of the function f. Having identified y with the output variable, we can interpret the equation $y = x + 1$ as the definition of the function $f(x) = x + 1$.

Note that the equation $x - 3 = y - 4$ and the function $f(x) = x + 1$ describe the same relationship between the variables x and y. For example, if we set the value $x = 5$ we can find the value of y by solving the equation $5 - 3 = y - 4$ to obtain $y = 6$, or by computing the output of the function $f(x)$ for the input $x = 5$, which gives us the same answer $f(5) = 6$. In both cases we arrive at the same answer, but modelling the relationship between x and y as a function allows us to use the whole functions toolbox, like function composition and function inverses.

In this section we talked a lot about functions in general but we haven't said much about any function specifically. There are many useful functions out there, and we can't discuss them all here. In the next section, we'll introduce 10 functions of strategic importance for all of science. If you get a grip on these functions, you'll be able to understand all of physics and calculus and handle *any* problem your teacher may throw at you.

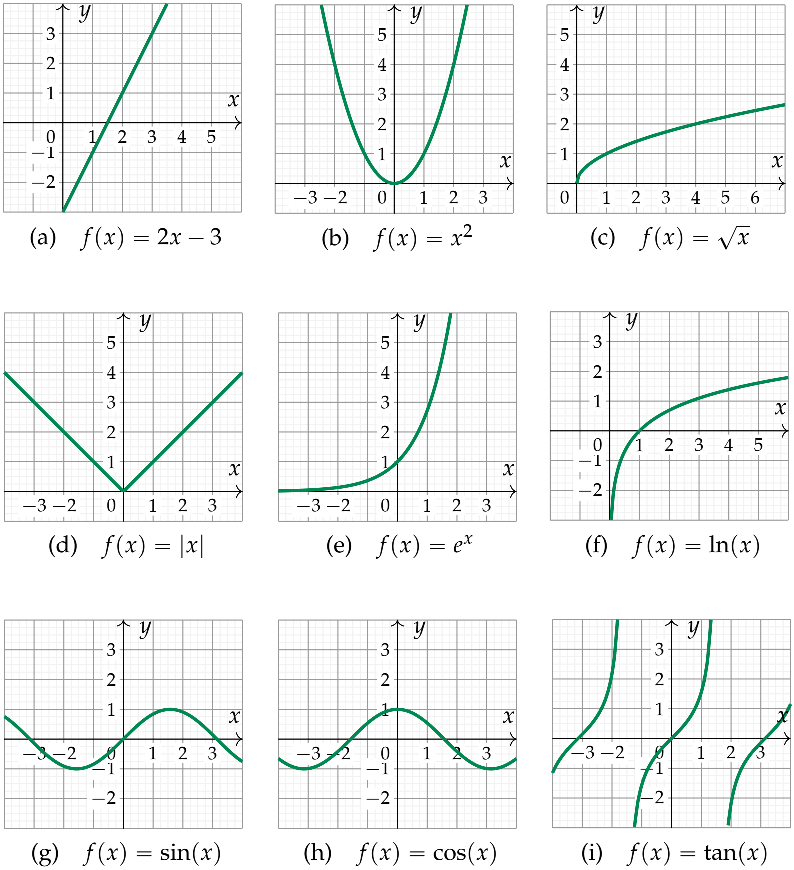
5.2 Functions reference

Your *function vocabulary* determines how well you can express yourself mathematically in the same way your English vocabulary determines how well you can express yourself in English. The following pages aim to embiggen your function vocabulary, so you'll know how to handle the situation when a teacher tries to pull some trick on you at the final. Here are the ten most important functions in math:

1. Straight line $f(x) = mx + b$ (see pages 71 and 165)
2. Quadratic function $f(x) = x^2$ (pages 73, 90, and 120)
3. Square root $f(x) = \sqrt{x}$ (page 74)
4. Absolute value $f(x) = |x|$ (page 75)
5. Polynomials $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ (page 76)
6. Sine $f(x) = \sin(x)$ (pages 80, 91, and 99)
7. Cosine $f(x) = \cos(x)$ (pages 82 and 99)
8. Tangent $f(x) = \tan(x)$ (page 83)
9. Exponential $f(x) = e^x$ (pages 84 and 170)
10. Logarithm $f(x) = \ln(x)$ (page 85)

If you're seeing these functions for the first time, don't worry about remembering all the facts and properties on the first reading. We'll use these functions throughout the rest of the book, so you'll have plenty of time to become familiar with them. Remember to return to this section if you ever get stuck on a function.

To build mathematical intuition, it's essential you understand functions' graphs. Memorizing the definitions and properties of functions gets a lot easier with visual accompaniment. Indeed, remembering what the function "looks like" is a great way to train yourself to recognize various types of functions. Figure 5.9 shows the graphs of some of the functions we'll use in this book.



(a) $f(x) = 2x - 3$ (b) $f(x) = x^2$ (c) $f(x) = \sqrt{x}$

(d) $f(x) = |x|$ (e) $f(x) = e^x$ (f) $f(x) = \ln(x)$

(g) $f(x) = \sin(x)$ (h) $f(x) = \cos(x)$ (i) $f(x) = \tan(x)$

Figure 5.9: We'll see many types of function graphs in the next pages.

Line

The equation of a line describes an input-output relationship where the change in the output is *proportional* to the change in the input. The equation of a line is

$$f(x) = mx + b.$$

The constant m describes the slope of the line. The constant b is called the y-intercept and it is the value of the function when $x = 0$.

Consider what relationship the equation of $f(x)$ describes for different values of m and b. What happens when m is positive? What happens when m is negative?

Graph

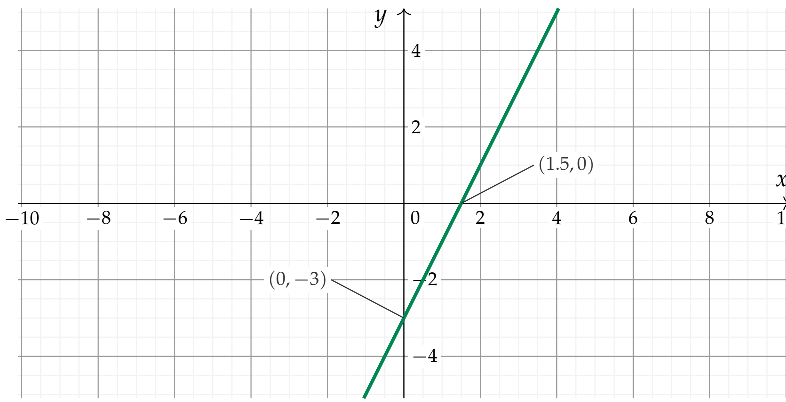


Figure 5.10: The graph of the function $f(x) = 2x - 3$. The slope is $m = 2$. The y-intercept of this line is $b = -3$. The x-intercept is at $x = \frac{3}{2}$.

Properties

- Domain: $\mathbb{R}$. The function $f(x) = mx + b$ is defined for all reals.
- Image: $\mathbb{R}$ if $m \neq 0$. If $m = 0$ the function is constant $f(x) = b$, so the image set contains only a single number $\{b\}$.
- $x = -b/m$: the x-intercept of $f(x) = mx + b$. The x-intercept is obtained by solving $f(x) = 0$.
- The inverse to the line $f(x) = mx + b$ is $f^{-1}(x) = \frac{1}{m}(x - b)$, which is also a line.

General equation

A line can also be described in a more symmetric form as a relation:

$$Ax + By = C.$$

This is known as the *general* equation of a line. The general equation for the line shown in Figure 5.10 is $2x - 1y = 3$.

Given the general equation of a line $Ax + By = C$ with $B \neq 0$, you can convert to the function form $y = f(x) = mx + b$ by computing the slope $m = \frac{-A}{B}$ and the y-intercept $b = \frac{C}{B}$.

Square

The function x *squared*, is also called the *quadratic* function, or *parabola*. The formula for the quadratic function is

$$f(x) = x^2.$$

The name "quadratic" comes from the Latin *quadratus* for square, since the expression for the area of a square with side length x is x^2.

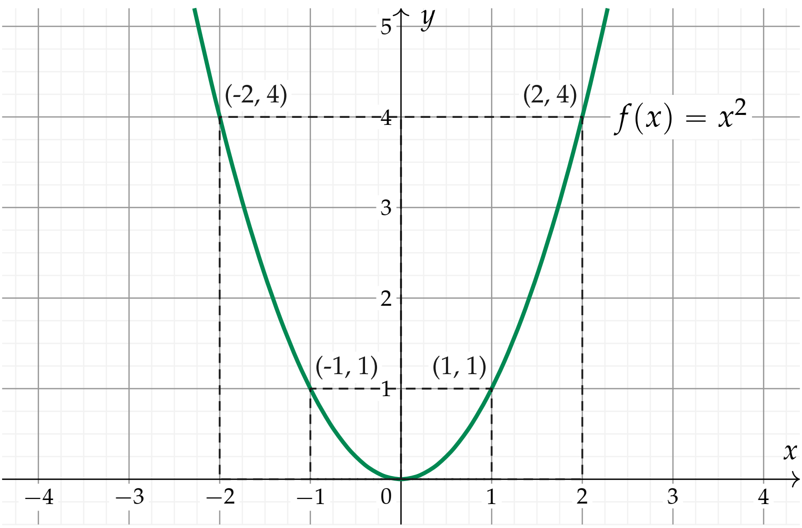


Figure 5.11: Plot of the quadratic function $f(x) = x^2$. The graph of the function passes through the following (x, y) coordinates: $(-2, 4)$, $(-1, 1)$, $(0, 0)$, $(1, 1)$, $(2, 4)$, $(3, 9)$, etc.

Properties

- Domain: $\mathbb{R}$. The function $f(x) = x^2$ is defined for all numbers.
- Image: $\mathbb{R}_+ = \{y \in \mathbb{R} \mid y \geqslant 0\}$. The outputs are nonnegative numbers since $x^2 \geqslant 0$, for all real numbers x.
- The function x^2 is the inverse of the square root function $\sqrt{x}$.
- $f(x) = x^2$ is *two-to-one*: it sends both x and $-x$ to the same output value $x^2 = (-x)^2$.
- The quadratic function is *convex*, meaning it curves upward.

The set expression $\{y \in \mathbb{R} \mid y \geqslant 0\}$ that we use to define the non-negative real numbers ($\mathbb{R}_+$) is read "the set of real numbers that are greater than or equal to zero."

Square root

The square root function is denoted

$$f(x) = \sqrt{x} = x^{\frac{1}{2}}.$$

The square root $\sqrt{x}$ is the inverse function of the square function x^2 when the two functions are defined as $f : \mathbb{R}_+ \to \mathbb{R}_+$. The symbol $\sqrt{c}$ refers to the *positive* solution of $x^2 = c$. Note that $-\sqrt{c}$ is also a solution of $x^2 = c$.

Graph

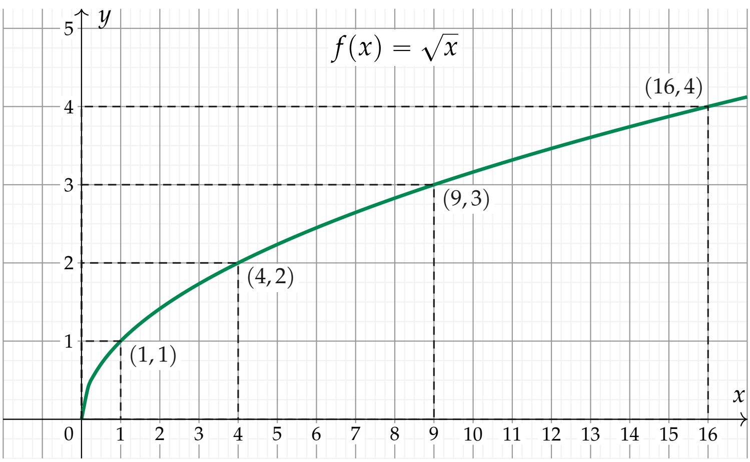


Figure 5.12: The graph of the function $f(x) = \sqrt{x}$. The domain of the function is $\mathbb{R}_+$ because we can't take the square root of a negative number.

Properties

- Domain: $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geqslant 0\}$. The function $f(x) = \sqrt{x}$ is only defined for nonnegative inputs. There is no real number y such that y^2 is negative, hence the function $f(x) = \sqrt{x}$ is not defined for negative inputs x.
- Image: $\mathbb{R}_+ = \{y \in \mathbb{R} \mid y \geqslant 0\}$. The outputs of the function $f(x) = \sqrt{x}$ are nonnegative numbers since $\sqrt{x} \geqslant 0$.

In addition to *square* root, there is also *cube* root $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$, which is the inverse function for the cubic function $f(x) = x^3$. We have $\sqrt[3]{8} = 2$ since $2 \times 2 \times 2 = 8$. More generally, we can define the n^{th}-root function $\sqrt[n]{x}$ as the inverse function of x^n.

Absolute value

The *absolute value* function tells us the size of numbers without paying attention to whether the number is positive or negative. We can compute a number's absolute value by *ignoring the sign* of the number. A number's absolute value corresponds to its distance from the origin of the number line.

Another way of thinking about the absolute value function is to say it multiplies negative numbers by -1 to "cancel" their negative sign:

$$f(x) = |x| = \begin{cases} x & \text{if } x \geqslant 0, \\ -x & \text{if } x < 0. \end{cases}$$

Graph

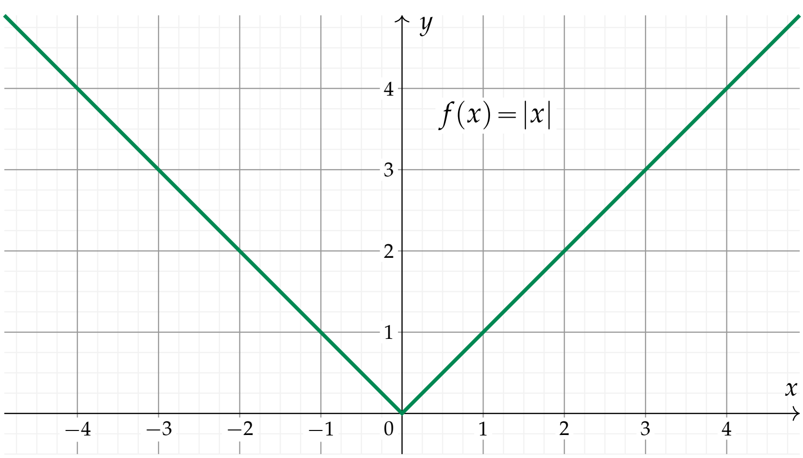


Figure 5.13: The graph of the absolute value function $f(x) = |x|$.

Properties

- Domain: $\mathbb{R}$. The function $f(x) = |x|$ is defined for all inputs.
- Image: $\mathbb{R}_+ = \{y \in \mathbb{R} \mid y \geqslant 0\}$
- The combination of squaring followed by square-root is equivalent to the absolute value function:

$$\sqrt{x^2} = |x|,$$

since squaring destroys the sign.

Polynomials

The polynomials are a very useful family of functions. For example, quadratic polynomials of the form $f(x) = ax^2 + bx + c$ often arise when describing physics phenomena.

The general equation for a polynomial function of degree n is

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n.$$

The constants a_i are known as the *coefficients* of the polynomial.

Parameters

- x: the variable
- a_0: the constant term
- a_1: the *linear* coefficient, or *first-order* coefficient
- a_2: the *quadratic* coefficient
- a_3: the *cubic* coefficient
- a_n: the n^{th} order coefficient
- n: the *degree* of the polynomial. The degree of $f(x)$ is the largest power of x that appears in the polynomial.

A polynomial of degree n has $n+1$ coefficients: $a_0, a_1, a_2, \ldots, a_n$.

Properties

- Domain: $\mathbb{R}$. Polynomials are defined for all inputs.
- The roots of $f(x)$ are the values of x for which $f(x) = 0$.
- The image of a polynomial function depends on the coefficients.
- The sum of two polynomials is also a polynomial.

The most general first-degree polynomial is a line $f(x) = mx + b$, where m and b are arbitrary constants. The most general second-degree polynomial is $f(x) = a_2 x^2 + a_1 x + a_0$, where again a_0, a_1, and a_2 are arbitrary constants. We call a_k the *coefficient* of x^k, since this is the number that appears in front of x^k. Following the pattern, a third-degree polynomial will look like $f(x) = a_3 x^3 + a_2 x^2 + a_1 x + a_0$.

In general, a polynomial of degree n has the equation

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0.$$

You can add two polynomials by adding together their coefficients:

$$\begin{aligned} f(x) + g(x) &= (a_n x^n + \cdots + a_1 x + a_0) \ + \ (b_n x^n + \cdots + b_1 x + b_0) \\ &= (a_n + b_n) x^n + \cdots + (a_1 + b_1) x + (a_0 + b_0). \end{aligned}$$

The subtraction of two polynomials works similarly. We can also multiply polynomials together using the general algebra rules for expanding brackets.

Solving polynomial equations

Very often in math, you will have to *solve* polynomial equations of the form

$$A(x) = B(x),$$

where $A(x)$ and $B(x)$ are both polynomials. Recall from earlier that to *solve*, we must find the values of x that make the equality true.

Say the revenue of your company is a function of the number of products sold x, and can be expressed as $R(x) = 2x^2 + 2x$. Say also the cost you incur to produce x objects is $C(x) = x^2 + 5x + 10$. You want to determine the amount of product you need to produce to break even, that is, so that revenue equals cost: $R(x) = C(x)$. To find the break-even value x, solve the equation

$$2x^2 + 2x = x^2 + 5x + 10.$$

This may seem complicated since there are xs all over the place. No worries! We can turn the equation into its "standard form," and then use the quadratic formula. First, move all the terms to one side until only zero remains on the other side:

$$\begin{aligned} 2x^2 + 2x \quad - x^2 &= \cancel{x^2} + 5x + 10 \quad - \cancel{x^2} \\ x^2 + 2x \quad - 5x &= \cancel{5x} + 10 \quad - \cancel{5x} \\ x^2 - 3x \quad - 10 &= \cancel{10} \quad - \cancel{10} \\ x^2 - 3x - 10 &= 0. \end{aligned}$$

Remember, if we perform the same operations on both sides of the equation, the resulting equation has the same solutions. Therefore, the values of x that satisfy $x^2 - 3x - 10 = 0$, namely $x = -2$ and $x = 5$, also satisfy $2x^2 + 2x = x^2 + 5x + 10$, which is the original problem we're trying to solve.

This "shuffling of terms" approach will work for any polynomial equation $A(x) = B(x)$. We can always rewrite it as $C(x) = 0$, where $C(x)$ is a new polynomial with coefficients equal to the difference of the coefficients of A and B. Don't worry about which side you move all the coefficients to because $C(x) = 0$ and $0 = -C(x)$ have exactly the same solutions. Furthermore, the degree of the polynomial C can be no greater than that of A or B.

The form $C(x) = 0$ is the *standard form* of a polynomial, and we'll explore several formulas you can use to find its solution(s).

Formulas

The formula for solving the polynomial equation $P(x) = 0$ depends on the *degree* of the polynomial in question.

For a first-degree polynomial equation, $P_1(x) = mx + b = 0$, the solution is $x = \frac{-b}{m}$: just move b to the other side and divide by m.

For a second-degree polynomial,

$$P_2(x) = ax^2 + bx + c = 0,$$

the solutions are $x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$.

If $b^2 - 4ac < 0$, the solutions will involve taking the square root of a negative number. In those cases, we say no real solutions exist.

There is also a formula for polynomials of degree 3 and 4, but they are complicated. For polynomials with order $\geqslant 5$, there does not exist a general analytical solution.

Using a computer

When solving real-world problems, you'll often run into much more complicated equations. To find the solutions of anything more complicated than the quadratic equation, I recommend using a computer algebra system like `SymPy`: `http://live.sympy.org`.

To make `SymPy` solve the standard-form equation $C(x) = 0$, call the function `solve(expr,var)`, where the expression `expr` corresponds to $C(x)$, and `var` is the variable you want to solve for. For example, to solve $x^2 - 3x + 2 = 0$, type in the following:

```
>>> solve(x**2 - 3*x + 2, x)          # usage: solve(expr, var)
[1, 2]
```

The function `solve` will find the solutions to any equation of the form `expr = 0`. In this case, we see the solutions are $x = 1$ and $x = 2$.

Another way to solve the equation is to factor the polynomial $C(x)$ using the function `factor` like this:

```
>>> factor(x**2 - 3*x + 2)            # usage: factor(expr)
(x - 1)*(x - 2)
```

We see that $x^2 - 3x + 2 = (x - 1)(x - 2)$, which confirms the two roots are indeed $x = 1$ and $x = 2$.

To learn more about `SymPy`, check out Appendix C on page 217, which talks about all the `SymPy` functions that are available to you.

Substitution trick

Sometimes you can solve fourth-degree polynomials by using the quadratic formula. Say you're asked to solve for x in

$$x^4 - 7x^2 + 10 = 0.$$

Imagine this problem is on your exam, where you are not allowed to use a computer. How does the teacher expect you to solve for x? The trick is to substitute $y = x^2$ and rewrite the same equation as

$$y^2 - 7y + 10 = 0,$$

which you can solve by applying the quadratic formula. If you obtain the solutions $y = \alpha$ and $y = \beta$, then the solutions to the original fourth-degree polynomial are $x = \pm\sqrt{\alpha}$ and $x = \pm\sqrt{\beta}$, since $y = x^2$.

Since we're not taking an exam right now, we are allowed to use the computer to find the roots:

```
>>> solve(y**2 - 7*y + 10, y)
[2, 5]
>>> solve(x**4 - 7*x**2 + 10, x)
[sqrt(2), -sqrt(2), sqrt(5), -sqrt(5)]
```

Note how the second-degree polynomial has two roots, while the fourth-degree polynomial has four roots.

Even and odd functions

The polynomials form an entire family of functions. Depending on the choice of degree n and coefficients a_0, a_1, ..., a_n, a polynomial function can take on many different shapes. Consider the following observations about the symmetries of polynomials:

- If a polynomial contains only even powers of x, like $f(x) = 1 + x^2 - x^4$ for example, we call this polynomial *even*. Even polynomials have the property $f(x) = f(-x)$. The sign of the input doesn't matter.
- If a polynomial contains only odd powers of x, for example $g(x) = x + x^3 - x^9$, we call this polynomial *odd*. Odd polynomials have the property $g(x) = -g(-x)$.
- If a polynomial has both even and odd terms then it is neither even nor odd.

The terminology of *odd* and *even* applies to functions in general and not just to polynomials. All functions that satisfy $f(x) = f(-x)$ are called *even functions*, and all functions that satisfy $f(x) = -f(-x)$ are called *odd functions*.

Sine

The sine function represents a fundamental unit of vibration. The graph of $\sin(x)$ *oscillates* up and down and crosses the x-axis multiple times. The shape of the graph of $\sin(x)$ corresponds to the shape of a vibrating string. See Figure 5.14.

In the remainder of this book, we'll meet the function $\sin(x)$ many times. We'll define the function $\sin(x)$ more formally as a trigonometric ratio in Section 6.2. In Chapter 7 we'll use $\sin(x)$ and $\cos(x)$ (another trigonometric ratio) to work out the *components* of vectors. The sine function also describes waves and periodic motion.

At this point in the book, however, we don't want to go into too much detail about all these applications. Let's hold off on the discussion about vectors, triangles, angles, and ratios of lengths of sides and instead just focus on the graph of the function $f(x) = \sin(x)$.

Graph

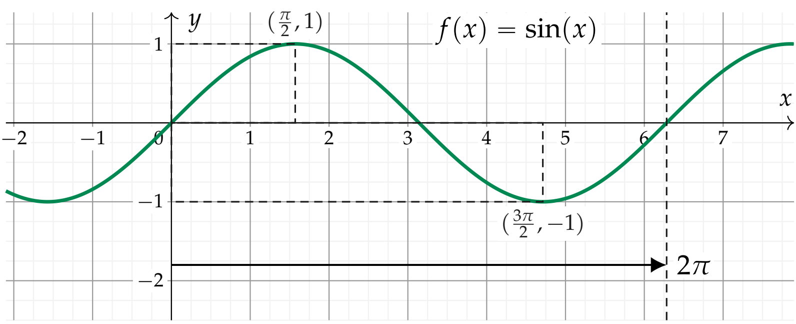


Figure 5.14: The graph of the function $y = \sin(x)$ passes through the following (x, y) coordinates: $(0, 0)$, $(\frac{\pi}{6}, \frac{1}{2})$, $(\frac{\pi}{4}, \frac{\sqrt{2}}{2})$, $(\frac{\pi}{3}, \frac{\sqrt{3}}{2})$, $(\frac{\pi}{2}, 1)$, $(\frac{2\pi}{3}, \frac{\sqrt{3}}{2})$, $(\frac{3\pi}{4}, \frac{\sqrt{2}}{2})$, $(\frac{5\pi}{6}, \frac{1}{2})$, and $(\pi, 0)$. For x between π and 2π, the function's graph has the same shape it has for x between 0 and π, but with negative values.

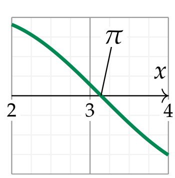


Figure 5.15: The function $f(x) = \sin(x)$ crosses the x-axis at $x = \pi$.

Let's start at $x = 0$ and follow the graph of the function $\sin(x)$ as it goes up and down. The graph starts from $(0, 0)$ and smoothly increases until it reaches the maximum value at $x = \frac{\pi}{2}$. Afterward, the function comes back down to cross the x-axis at $x = \pi$. After π, the function drops below the x-axis and reaches its minimum value of -1 at $x = \frac{3\pi}{2}$. It then travels up again to cross the x-axis at $x = 2\pi$. This 2π-long cycle repeats after $x = 2\pi$. This is why we call the function *periodic*—the shape of the graph repeats.

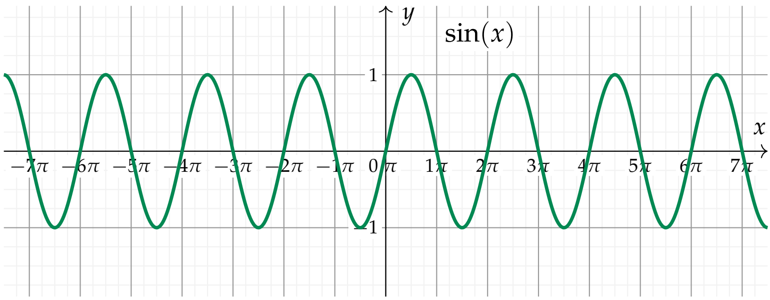


Figure 5.16: The graph of $\sin(x)$ from $x = 0$ to $x = 2\pi$ repeats periodically everywhere else on the number line.

Properties

- Domain: $\mathbb{R}$. The function $f(x) = \sin(x)$ is defined for all input values.
- Image: $\{y \in \mathbb{R} \mid -1 \leqslant y \leqslant 1\}$. The outputs of the sine function are always between -1 and 1.
- Roots: $\{\ldots, -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \ldots\}$.
  The function $\sin(x)$ has roots at all multiples of π.
- The function is periodic, with period 2π: $\sin(x) = \sin(x + 2\pi)$.
- The sin function is *odd*: $\sin(x) = -\sin(-x)$
- Relation to cos: $\sin^2 x + \cos^2 x = 1$
- Relation to csc: $\csc(x) = \frac{1}{\sin x}$ (csc is read *cosecant*)
- The inverse function of $\sin(x)$ is denoted as $\sin^{-1}(x)$ or $\arcsin(x)$, not to be confused with $(\sin(x))^{-1} = \frac{1}{\sin(x)} = \csc(x)$.
- The number $\sin(\theta)$ is the length-ratio of the vertical side and the hypotenuse in a right-angle triangle with angle θ at the base.

Links

[See the Wikipedia page for nice illustrations]
`http://en.wikipedia.org/wiki/Sine`

Cosine

The cosine function is the same as the sine function *shifted* by $\frac{\pi}{2}$ to the left: $\cos(x) = \sin(x + \frac{\pi}{2})$. Thus everything you know about the sine function also applies to the cosine function.

Graph

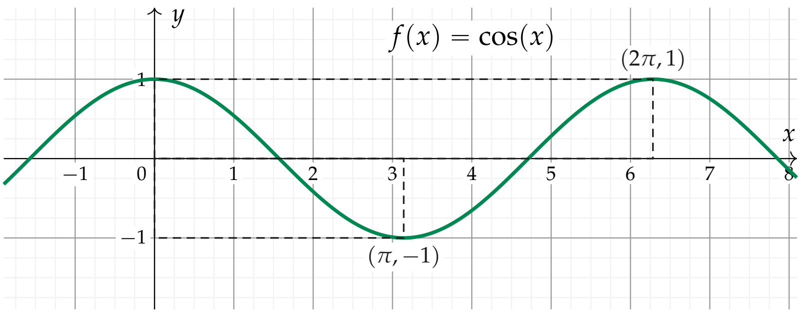


Figure 5.17: The graph of the function $y = \cos(x)$ passes through the following (x, y) coordinates: $(0, 1)$, $(\frac{\pi}{6}, \frac{\sqrt{3}}{2})$, $(\frac{\pi}{4}, \frac{\sqrt{2}}{2})$, $(\frac{\pi}{3}, \frac{1}{2})$, $(\frac{\pi}{2}, 0)$, $(\frac{2\pi}{3}, -\frac{1}{2})$, $(\frac{3\pi}{4}, -\frac{\sqrt{2}}{2})$, $(\frac{5\pi}{6}, -\frac{\sqrt{3}}{2})$, and $(\pi, -1)$.

The cos function starts at $\cos(0) = 1$, then drops down to cross the x-axis at $x = \frac{\pi}{2}$. Cos continues until it reaches its minimum value at $x = \pi$. The function then moves upward, crossing the x-axis again at $x = \frac{3\pi}{2}$, and reaching its maximum value again at $x = 2\pi$.

Properties

- Domain: $\mathbb{R}$
- Image: $\{y \in \mathbb{R} \mid -1 \leqslant y \leqslant 1\}$
- Roots: $\{\ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots\}$
- Relation to sin: $\sin^2 x + \cos^2 x = 1$
- Relation to sec: $\sec(x) = \frac{1}{\cos x}$ (sec is read *secant*)
- The inverse function of $\cos(x)$ is denoted $\cos^{-1}(x)$ or $\arccos(x)$.
- The cos function is *even*: $\cos(x) = \cos(-x)$
- The number $\cos(\theta)$ is the length-ratio of the horizontal side and the hypotenuse in a right-angle triangle with angle θ at the base

Tangent

The tangent function is the ratio of the sine and cosine functions:

$$f(x) = \tan(x) = \frac{\sin(x)}{\cos(x)}.$$

Graph

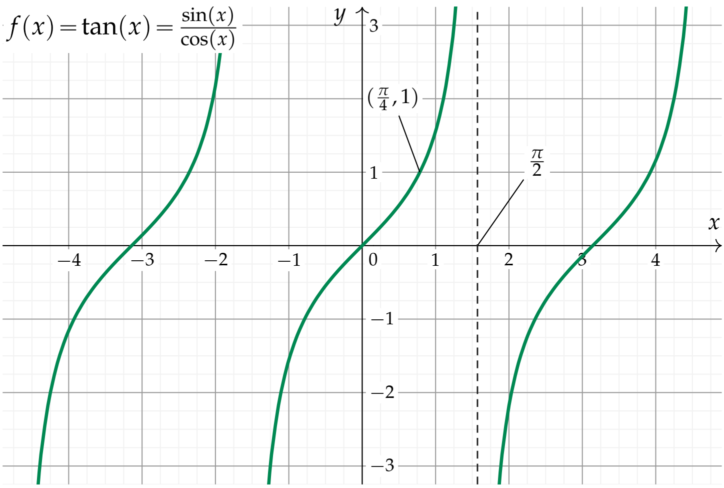


Figure 5.18: The graph of the function $f(x) = \tan(x)$.

Properties

- Domain: $\{x \in \mathbb{R} \mid x \neq \frac{(2n+1)\pi}{2} \text{ for any } n \in \mathbb{Z}\}$
- Image: $\mathbb{R}$
- The function tan is periodic with period π.
- The tan function "blows up" at values of x where $\cos x = 0$. These are called *asymptotes* of the function and their locations are $x = \ldots, \frac{-3\pi}{2}, \frac{-\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$.
- Value at $x = 0$: $\tan(0) = \frac{0}{1} = 0$, because $\sin(0) = 0$.
- Value at $x = \frac{\pi}{4}$: $\tan\left(\frac{\pi}{4}\right) = \frac{\sin(\frac{\pi}{4})}{\cos(\frac{\pi}{4})} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$.
- The number $\tan(\theta)$ is the length-ratio of the vertical and the horizontal sides in a right-angle triangle with angle θ.
- The inverse function of $\tan(x)$ is denoted $\tan^{-1}(x)$ or $\arctan(x)$.
- The inverse tangent function is used to compute the angle at the base in a right-angle triangle with horizontal side length ℓ_h and vertical side length ℓ_v: $\theta = \tan^{-1}\left(\frac{\ell_v}{\ell_h}\right)$.

Exponential

The exponential function base $e = 2.7182818\ldots$ is denoted

$$f(x) = e^x = \exp(x).$$

Graph

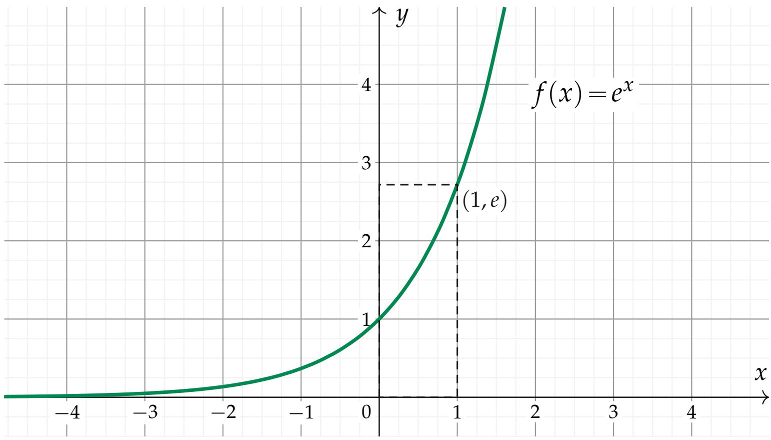


Figure 5.19: The graph of the exponential function $f(x) = e^x$ passes through the following points: $(-2, \frac{1}{e^2})$, $(-1, \frac{1}{e})$, $(0, 1)$, $(1, e)$, $(2, e^2)$, $(3, e^3)$, $(4, e^4)$, etc.

Properties

- Domain: $\mathbb{R}$
- Image: $\{y \in \mathbb{R} \mid y > 0\}$
- $f(a)f(b) = f(a+b)$ since $e^a e^b = e^{a+b}$

A more general exponential function would be $f(x) = Ae^{\gamma x}$, where A is the initial value, and γ (the Greek letter *gamma*) is the *rate* of the exponential. For $\gamma > 0$, the function $f(x)$ is increasing, as in Figure 5.19. For $\gamma < 0$, the function is decreasing and tends to zero for large values of x. The case $\gamma = 0$ is special since $e^0 = 1$, so $f(x)$ is a constant of $f(x) = A1^x = A$.

Links

[The exponential function 2^x evaluated]
`http://www.youtube.com/watch?v=e4MSN6IImpI`

Natural logarithm

The natural logarithm function is denoted

$$f(x) = \ln(x) = \log_e(x).$$

The function $\ln(x)$ is the inverse function of the exponential e^x.

Graph

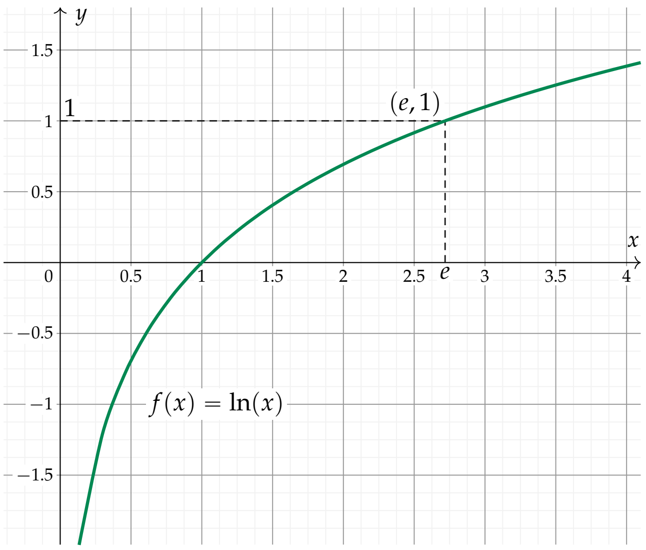


Figure 5.20: The graph of the function $\ln(x)$ passes through the following coordinates: $(\frac{1}{e^2}, -2)$, $(\frac{1}{e}, -1)$, $(1, 0)$, $(e, 1)$, $(e^2, 2)$, $(e^3, 3)$, $(e^4, 4)$, etc.

Properties

- Domain: $\{x \in \mathbb{R} \mid x > 0\}$
- Image: $\mathbb{R}$

Exercises

E5.1 Find the domain, the image, and the roots of $f(x) = 2\cos(x)$.

E5.2 What are the degrees of the following polynomials? Are they even, odd, or neither?

a) $p(x) = x^2 - 5x^4 + 1$ **b)** $q(x) = x - x^3 + x^5 - x^7$

E5.3 Solve for x in the following polynomial equations.

a) $3x + x^2 = x - 15 + 2x^2$ **b)** $3x^2 - 4x - 4 + x^3 = x^3 + 2x + 2$

5.3 Function transformations

Often, we're asked to adjust the shape of a function by scaling it or moving it, so that it passes through certain points. For example, if we wanted to make a function g with the same shape as the absolute value function $f(x) = |x|$, but moved up by three units so that $g(0) = 3$, we would use the function $g(x) = |x| + 3$.

In this section, we'll discuss the four basic transformations you can perform on *any* function f to obtain a transformed function g:

- Vertical translation: $g(x) = f(x) + k$
- Horizontal translation: $g(x) = f(x - h)$
- Vertical scaling: $g(x) = Af(x)$
- Horizontal scaling: $g(x) = f(ax)$

By applying these transformations, we can *move* and *stretch* a generic function to give it any desired shape.

The next couple of pages illustrate all of the above transformations on the function

$$f(x) = 6.75(x^3 - 2x^2 + x).$$

We'll work with this function because it has distinctive features in both the horizontal and vertical directions. By observing this function's graph (Figure 5.21), we see its x-intercepts are at $x = 0$ and $x = 1$. We can confirm this mathematically by factoring the expression:

$$f(x) = 6.75x(x^2 - 2x + 1) = 6.75x(x - 1)^2.$$

The function $f(x)$ also has a local maximum at $x = \frac{1}{3}$, and the value of the function at that maximum is $f(\frac{1}{3}) = 1$.

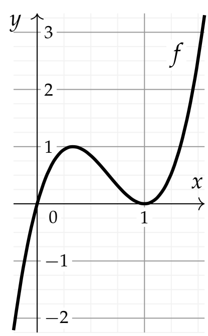


Figure 5.21: Graph of the function $f(x) = 6.75(x^3 - 2x^2 + x)$.

Vertical translations

To move a function $f(x)$ *up* by k units, add k to the function:

$$g(x) = f(x) + k.$$

The function $g(x)$ will have exactly the same shape as $f(x)$, but it will be *translated* (the mathematical term for moved) upward by k units.

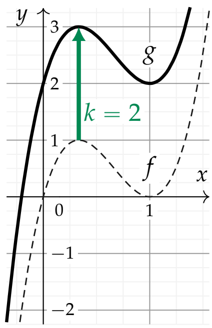


Figure 5.22: The graph of the function $g(x) = f(x) + 2$ has the same shape as the graph of $f(x)$ translated upward by two units.

Recall the function $f(x) = 6.75(x^3 - 2x^2 + x)$. To move the function up by $k = 2$ units, we can write

$$g(x) = f(x) + 2 = 6.75(x^3 - 2x^2 + x) + 2,$$

and the graph of $g(x)$ will be as it is shown in Figure 5.22. Recall the original function $f(x)$ crosses the x-axis at $x = 0$. The transformed function $g(x)$ has the property $g(0) = 2$. The maximum at $x = \frac{1}{3}$ has similarly shifted in value from $f(\frac{1}{3}) = 1$ to $g(\frac{1}{3}) = 3$.

Horizontal translation

We can move a function f to the right by h units by *subtracting* h from x and using $(x - h)$ as the function's input argument:

$$g(x) = f(x - h).$$

The point $(0, f(0))$ on $f(x)$ now corresponds to the point $(h, g(h))$ on $g(x)$.

Figure 5.23 shows the function $f(x) = 6.75(x^3 - 2x^2 + x)$, as well as the function $g(x)$, which is shifted to the right by $h = 2$ units:

$$g(x) = f(x - 2) = 6.75\left[(x-2)^3 - 2(x-2)^2 + (x-2)\right].$$

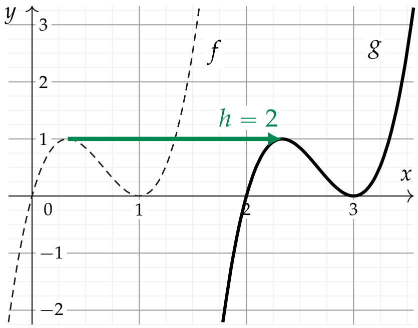


Figure 5.23: The graph of the function $g(x) = f(x-2)$ has the same shape as the graph of $f(x)$ translated to the right by two units.

The original function f gives us $f(0) = 0$ and $f(1) = 0$, so the new function $g(x)$ must give $g(2) = 0$ and $g(3) = 0$. The maximum at $x = \frac{1}{3}$ has similarly shifted by two units to the right, $g(2 + \frac{1}{3}) = 1$.

Vertical scaling

To stretch or compress the shape of a function vertically, we can multiply it by some constant A and obtain

$$g(x) = Af(x).$$

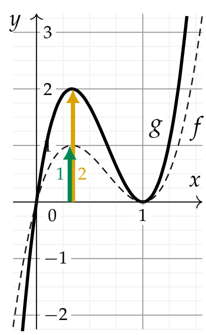


Figure 5.24: The graph of the function $g(x) = 2f(x)$ looks like $f(x)$ vertically stretched by a factor of two.

If $|A| > 1$, the function will be stretched. If $|A| < 1$, the function will be compressed. If A is negative, the function will flip upside down, which is a *reflection* through the x-axis.

There is an important difference between vertical translation and vertical scaling. Translation moves all points of the function by the same amount, whereas scaling moves each point proportionally to that point's distance from the x-axis.

The function $f(x) = 6.75(x^3 - 2x^2 + x)$, when stretched vertically by a factor of $A = 2$, becomes the function

$$g(x) = 2f(x) = 13.5(x^3 - 2x^2 + x).$$

The x-intercepts $f(0) = 0$ and $f(1) = 0$ do not move, and remain at $g(0) = 0$ and $g(1) = 0$. All values of $f(x)$ have been stretched upward by a factor of 2, as we can verify using the point $f(1.5) = 2.5$, which has become $g(1.5) = 5$. The maximum at $x = \frac{1}{3}$ has doubled in value to become $g(\frac{1}{3}) = 2$.

Horizontal scaling

To stretch or compress a function horizontally, we can multiply the input value by some constant a to obtain:

$$g(x) = f(ax).$$

If $|a| > 1$, the function will be compressed. If $|a| < 1$, the function will be stretched. Note that the behaviour here is the opposite of vertical scaling. If a is a negative number, the function will also flip horizontally, which is a reflection through the y-axis.

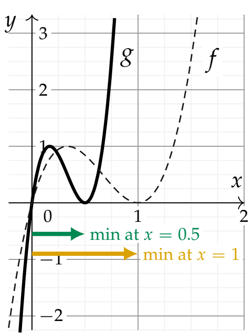


Figure 5.25: The graph of the function $g(x) = f(2x)$ looks like $f(x)$ horizontally compressed by a factor of two.

Figure 5.25 shows the function $g(x)$, which is $f(x)$ compressed horizontally by a factor of $a = 2$:

$$g(x) = f(2x) = 6.75\Big[(2x)^3 - 2(2x)^2 + (2x)\Big].$$

The x-intercept $f(0) = 0$ does not move since it is on the y-axis. The x-intercept $f(1) = 0$ does move, however, and we have $g(0.5) = 0$. The maximum at $x = \frac{1}{3}$ moves to $g(\frac{1}{6}) = 1$. All points of $f(x)$ are compressed toward the y-axis by a factor of 2.

General quadratic function

Any quadratic function can be written in the form:

$$f(x) = a(x-h)^2 + k,$$

where x is the input, and a, h, and k are parameters. This is called the *vertex form* of the quadratic function, and the coordinate pair (h, k) is called the *vertex* of the parabola. This equation can be obtained by starting from the basic quadratic function x^2 (see Figure 5.11) and applying three transformations: a horizontal translation by h units, a vertical scaling by a, and finally a vertical translation by k units.

Parameters

- a: the slope multiplier
  - ▷ The larger the absolute value of a, the steeper the slope.
  - ▷ If $a < 0$ (negative), the function opens downward.
- h: the horizontal displacement of the function. Note that subtracting a number inside the bracket $(\ \ )^2$ (positive h) makes the function go to the right.
- k: the vertical displacement of the function

The graph in Figure 5.26 illustrates a quadratic function with parameters $a = 1$, $h = 1$ (one unit shifted to the right), and $k = -2$ (two units shifted down).

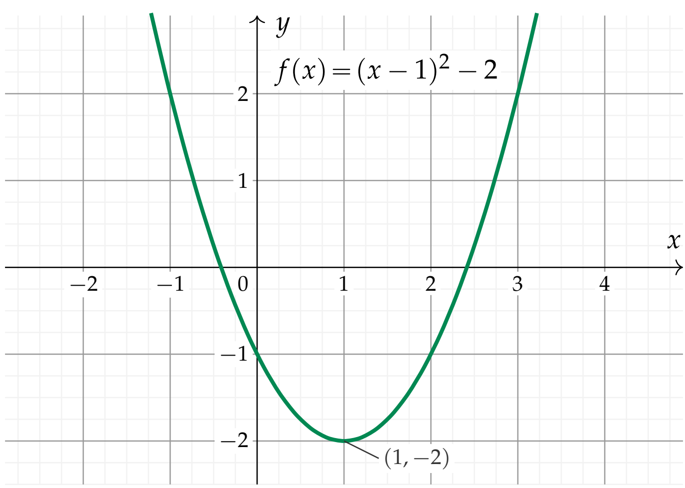


Figure 5.26: The graph of the function $f(x) = (x-1)^2 - 2$ is the same as the function $f(x) = x^2$, but shifted one unit to the right and two units down.

We can also write a quadratic function as a second-degree polynomial $f(x) = ax^2 + bx + c$. This is called the *standard form* of the quadratic function. Given a quadratic expression in standard form $ax^2 + bx + c$, we can find its equivalent expression in vertex form $a(x - h)^2 + k$ using the complete-the-square trick we learned in Section 2.1.

If a quadratic function crosses the x-axis, it can be written in *factored form*:

$$f(x) = a(x - x_1)(x - x_2),$$

where x_1 and x_2 are the two roots of the quadratic. Given a quadratic function $f(x) = ax^2 + bx + c$, we can find its roots using the quadratic formula: $x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$ (see Section 2.2).

General sine function

Introducing all possible parameters into the sine function gives us:

$$f(x) = A\sin\left(\tfrac{2\pi}{\lambda}x - \phi\right),$$

where A, λ, and ϕ are the function's parameters.

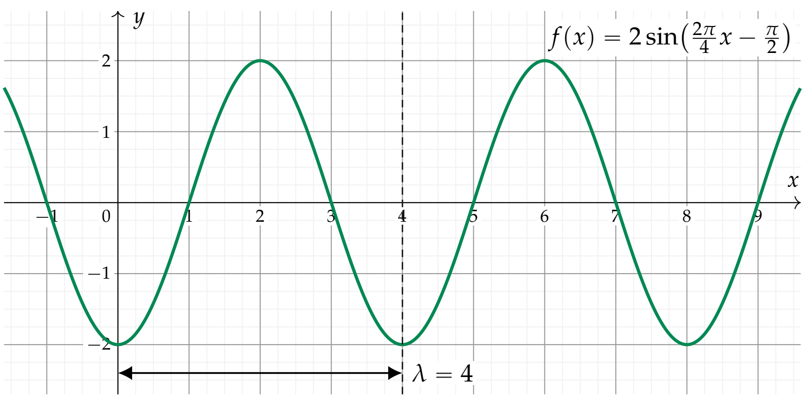


Figure 5.27: The graph of the function $f(x) = 2\sin\left(\frac{2\pi}{4}x - \frac{\pi}{2}\right)$, which has amplitude $A = 2$, wavelength $\lambda = 4$, and phase shift $\phi = \frac{\pi}{2}$.

Parameters

- A: the amplitude describes the distance above and below the x-axis that the function reaches as it oscillates.
- λ: the *wavelength* of the function:

$$\lambda = \{\text{ the horizontal distance from one peak to the next }\}.$$

- ϕ: is a phase shift, analogous to the horizontal shift h, which we have seen. This number dictates where the oscillation starts. The default sine function has zero phase shift ($\phi = 0$), so it passes through the origin with an increasing slope.

The "bare" sine function $f(x) = \sin(x)$ has wavelength 2π and produces outputs that oscillate between -1 and $+1$. When we multiply the bare function by the constant A, the oscillations will range between $-A$ and A. When the input x is scaled by the factor $\frac{2\pi}{\lambda}$, the wavelength of the function becomes λ.

Exercises

E5.4 Given the functions $f(x) = x + 5$, $g(x) = x - 6$, $h(x) = 7x$, and $q(x) = x^2$, find the formulas for the following composite functions:

a) $q \circ f$ **b)** $f \circ q$ **c)** $q \circ g$ **d)** $q \circ h$

In each case, describe how the graph of the composite function is related to the graph of $q(x)$.
Hint: Recall, "$\circ$" denotes function composition: $(f \circ g)(x) = f(g(x))$.

E5.5 Find the amplitude A, the wavelength λ, and the phase shift ϕ for the function $f(x) = 5\sin(62.83x - \frac{\pi}{8})$.

E5.6 Choose the coefficients a, b, and c for the quadratic function $f(x) = ax^2 + bx + c$ so that it passes through the points $(0, 5)$, $(1, 4)$, and $(2, 5)$.
Hint: Find the equation $f(x) = A(x - h)^2 + k$ first.

E5.7 Find the values α and β that will make the function $g(x) = 2\sqrt{x - \alpha} + \beta$ pass through the points $(3, -2)$, $(4, 0)$, and $(7, 2)$.

Chapter 5 solutions

Answers to exercises

E5.1 Domain: $\mathbb{R}$. Image: $[-2, 2]$. Roots: $\{\ldots, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots\}$. **E5.2 a)** $p(x)$ is even and has degree 4. **b)** $q(x)$ is odd and has degree 7. **E5.3 a)** $x = 5$ and $x = -3$; **b)** $x = 1 + \sqrt{3}$ and $x = 1 - \sqrt{3}$. **E5.4 a)** $(q \circ f)(x) = q(f(x)) = (x+5)^2$; $q(x)$ shifted five units to the left. **b)** $(f \circ q)(x) = x^2 + 5$; $q(x)$ shifted upward by five units. **c)** $(q \circ g)(x) = (x-6)^2$; $q(x)$ shifted six units to the right. **d)** $(q \circ h)(x) = 49x^2$; $q(x)$ horizontally compressed by a factor of seven. **E5.5** $A = 5$, $\lambda = 0.1$, and $\phi = \frac{\pi}{8}$. **E5.6** $f(x) = x^2 - 2x + 5$. **E5.7** $g(x) = 2\sqrt{x-3} - 2$.

Solutions to selected exercises

E5.3 **a)** Rewrite the equation putting all terms on the right-hand side: $0 = x^2 - 2x - 15$. We can factor this quadratic by inspection. Are there numbers a and b such that $a + b = -2$ and $ab = -15$? Yes, $a = -5$ and $b = 3$, so $0 = (x-5)(x+3)$. **b)** Rewrite the equation so all terms are on the left-hand side: $3x^2 - 6x - 6 = 0$. Nice, the cubic terms cancel! We'll use the quadratic formula to solve this equation $x = \frac{6 \pm \sqrt{(-6)^2 - 4(3)(-6)}}{6} = \frac{6 \pm 6\sqrt{3}}{6} = 1 \pm \sqrt{3}$.