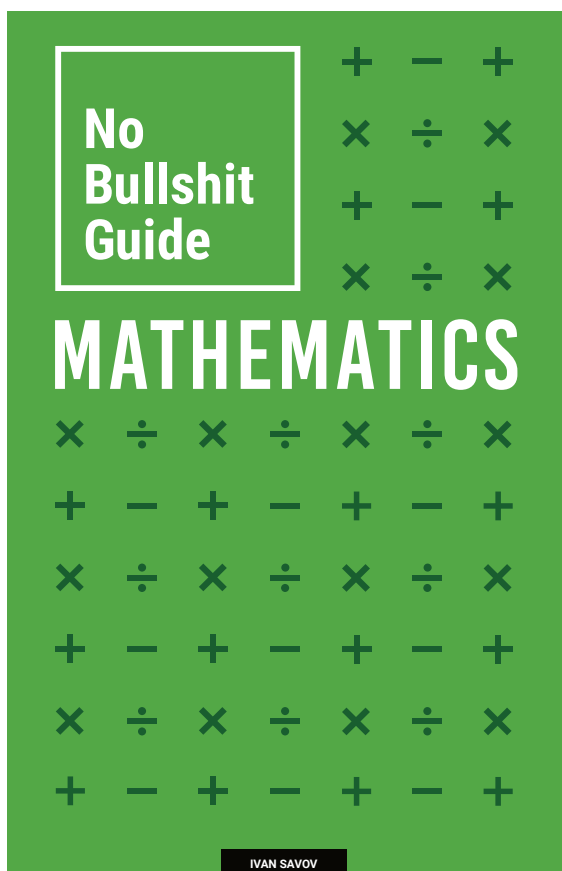


No Bullshit Guide to Mathematics

Extended book preview



The full book has 242 pages and includes 47 exercises and 83 solved problems. This preview has been chosen to showcase some key aspects of the book (chapter intros, formulas, explanations, and figures). You can buy the full book for only US\$10 at <https://gum.co/noBSmath>.

No Bullshit Guide to MATHEMATICS

Ivan Savov

October 25, 2020

No Bullshit Guide to Mathematics

by Ivan Savov

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Concept maps

The diagrams below show all the topics and concepts covered in this book. Topics are shown in rectangular boxes. Concepts are shown in bubbles with rounded corners. Note the numerous connections.

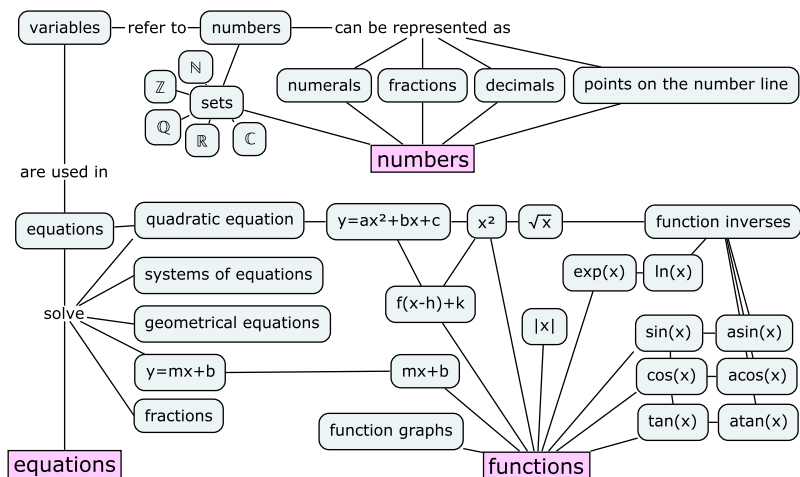


Figure 1: The first chapter of the book covers essential concepts like numbers, number representations, equations, and functions.

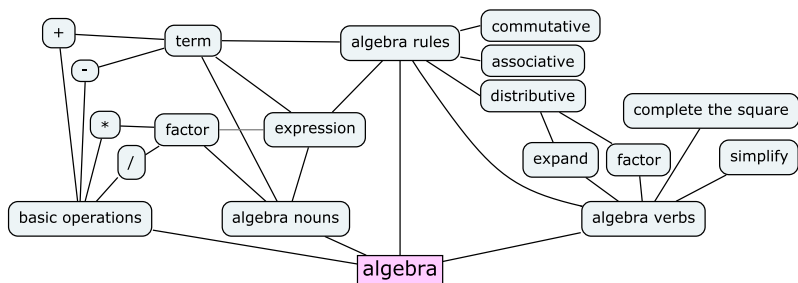


Figure 2: In Chapter 2 we'll learn about the rules of algebra, which codify what we're allowed to do with math expressions.

You can annotate the concept maps with your current knowledge of each concept to keep track of your progress. Add a single dot (●) next to all concepts you've heard of, two dots (●●) next to concepts you think you know, and three dots (●●●) next to concepts you've used in exercises and problems. By collecting some dots every week, you'll be able to move through the material in no time at all.

If you don't want to mark up your book, you can download a printable version of the concept maps here: bit.ly/mathcmap.

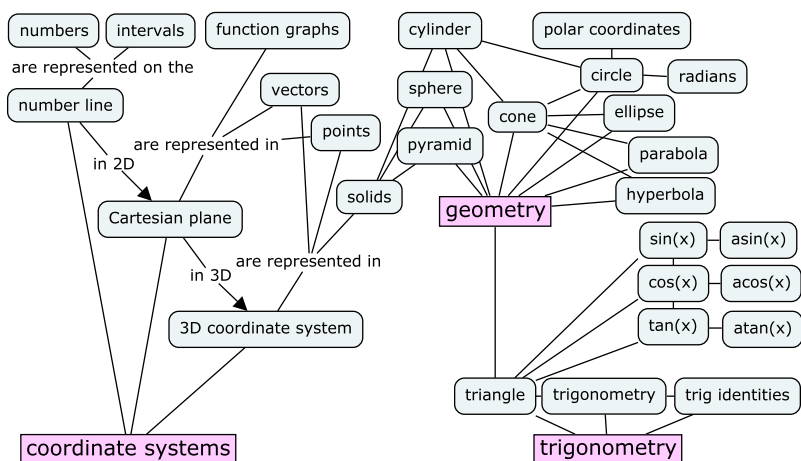


Figure 3: Visual thinking plays a very important role in mathematics. In Chapter 4 we'll learn about the Cartesian plane, which is a two-dimensional coordinate system used to represent points, vectors, and function graphs. Then in Chapter 6 we'll learn about the fundamental geometric shapes like triangles, pyramids, cones, circles, ellipses, parabolas, and hyperbolas. We'll also dedicate several sections to familiarizing ourselves with trigonometry, trigonometric functions, and trigonometric identities.

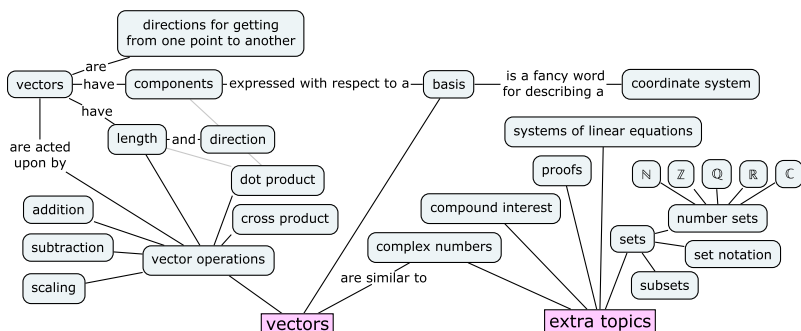


Figure 4: In Chapter 7, we'll use our algebra and geometry skills to learn about vectors and vector operations. In Chapter 8, we'll discuss math applications and formal math methods like proofs.

Consult the index on page 231 to find the exact location in the book where each concept is defined.

Preface

Why this book?

Math is a beautiful subject with many practical applications. Yet many people feel uncomfortable around math. Some even claim to “hate” math and try to avoid the subject. Countless adults are living with *math phobia*. That’s too bad because learning math opens many doors to understanding concepts in science, computing, music, and business. Math is also useful for accomplishing day-to-day tasks.

Why learn math?

Math is a universal language for describing patterns and shapes. Math is everywhere around us. When you learn math, you’ll be able to see the equations behind real-world phenomena, and then use these equations to predict the future. How cool is that? Learning math will also help you make better daily decisions like avoiding predatory loans, estimating times and distances when travelling, double-checking calculations in contracts and bills, and forecasting the cash flow of your business. Math is not just for scientists and engineers—*everyone* benefits from knowing math the same way everyone benefits from knowing how to read and write.

Is this book for you?

This book is written for adult learners who think they suck at math and feel uncomfortable around numbers and equations. Whether you hate math because of bad experiences, feel disheartened at the sight of equations, or simply don’t see the point of learning math, this book has something to offer you. It’s time to reconnect with this beautiful subject.

This book will be useful for everyone who wants to learn math. High school students can quickly learn the material they need for their classes. University students can use the book to review and prepare for more advanced subjects like calculus, statistics, and linear algebra. Parents who want to help their kids with homework can depend on this book to refresh their math skills.

How are we going to do this?

This book approaches math by showing you the connections between concepts. Learning math does not require memorizing facts and equations. It's not about how many equations you know, but about knowing how to get from one equation to another. Consult the concept maps on pages iii and iv to see all the topics covered in the book and how they are interconnected.

Each section in this book is a short tutorial that covers one math topic with precise definitions, illustrative examples, and concise explanations. By progressing through the chapters and sections of this book, you'll develop your math knowledge step by step, with each new layer of understanding building upon the previous ones. The book covers numbers, sets, variables, equations, algebra, functions, geometry, trigonometry, and also math applications to personal finance. If it's useful math, we've got it covered! To ensure that you *really* grasp the material, the book provides numerous exercises and practice problems (see Chapter 9). You'll find the answers to all the exercises and problems in Appendix A (page 205).

About the publisher

Minireference Co. is a publisher specializing in math and science. Our goal is to make advanced math modelling tools accessible to everyone by producing affordable textbooks that explain math concepts clearly and concisely. It's time for a break from traditional textbooks that are expensive, heavy, and tedious to read. The books in the **No Bullshit Guide** series are an efficient and enjoyable alternative for learning mathematical subjects. Whether you're a student, a parent, or an adult learner, we've got a book for you that will help you level-up in math.

About the author

I have been tutoring math and physics for more than 17 years. Through this experience, I learned to break complicated ideas into smaller, interconnected chunks that are easy to understand. I think the best way to teach math is to clearly define concepts and show the paths that connect them. When learners know the basic vocabulary and are able to find their way in a new mathematical domain, the battle against *math phobia* is already half won.

Ivan Savov
Montreal, 2020

Introduction

If you look carefully, you'll find mathematical patterns all around you: complex patterns like the water waves on the surface of a lake, medium-complexity patterns like the up-and-down oscillations of a mass suspended on a spring, and basic patterns that occur in all numbers and geometric shapes.

This is a book about basic math patterns. The goal is to introduce you to fundamental building blocks like numbers, expressions, functions, algebra, and geometry. Understanding basic math concepts will provide you with a common core of tools that you can use in diverse fields like physics, chemistry, biology, medicine, engineering, and business.

Thinking like a mathematician means being able to notice the patterns everywhere around you. That's what mathematicians do all day—seriously! First you *observe* a pattern; then you try to *describe* the pattern (this is usually where the equations and formulas come in); finally you *use* the pattern to achieve some practical outcome. Would you believe me if I told you that a few equations scribbled on a paper napkin can predict the outcome of a quantum physics experiment, calculate the yield of a chemical reaction, predict the growth rate of bacteria, save a patient's life, build a bridge, and maximize the profits of a startup?

Do I have your attention? I hope at this point you're interested in learning at least a little bit of math. Or perhaps you still feel kind of weird about reading a math book, and extra weird about reading a math book by choice, without someone forcing you to read it. What kind of trouble are you getting yourself into?

Relax. Math is no trouble. Learning math is not something to worry about or foresee as difficult. Math is not some sort of obstacle you have to jump over in one shot. Learning math is more like climbing a mountain. It's up to you to choose the pace of your climb. By investing some time in reading and solving exercises, everyone can gradually make it up the mountain.

The trick to learning math is to redefine what math means to you.

Dear reader, trust me on this one: math does not involve memorization and rote calculations! That's arithmetic. It's great if you're fast at adding, subtracting, multiplying, and dividing, but real math is so much more than that. For example, learning some basic algebra rules will allow you to simplify complicated expressions by rewriting them as equivalent but shorter expressions, so by the time you start plugging in the numbers you have a lot less arithmetic to do. The more algebra you learn, the less arithmetic you'll need to do. What? We learn more math in order to do less math? Yes, that's pretty much the whole point of the subject. Knowing how to recognize abstract patterns, how to describe them precisely, and how to use them to simplify problems turns out to be a very useful skill set.

In addition to being a useful tool for learning other sciences, learning math is interesting on its own. There will be some beautiful views at the end of each climb up the math mountains.

Guide to the math mountains

This book is your guide for the first part of your journey in the math mountains. Math concepts like numbers, equations, functions, algebra, and geometry are the first peaks you must climb on your way to the major math and science mountains.

In Chapter 1, we'll start by defining fundamental math ideas including numbers, variables, equations, and functions. These are the main building blocks of math, and we'll use them throughout the book. Readers who are new to mathematics should read Chapter 1 carefully and invest the time needed to solve the exercises.

Algebra! Yes, there, I said it. In Chapter 2 we'll learn the rules of algebra, which consist of various simplifications, reductions, and expansions we can perform on math expressions. Learning algebra by memorizing a bunch of rules is a very difficult task. Instead, you can think of algebra as a language. Algebra techniques are *verbs* that we apply to math expressions. To expand the expression $a(b + c)$ means to write the expression as $ab + ac$. To factor the expression $ax + ay + az$ is to write it as $a(x + y + z)$. To learn algebra you'll need to learn some new nouns: *variable*, *term*, *factor*, *expression*, *equation*. You'll also need to become familiar with the algebra verbs: *to solve* (equations), *to factor*, *to simplify*, *to expand* (expressions), *to complete the square* (in a quadratic expression), etc.

Are you still with me, dear reader? I know the prospects of learning algebra might not seem like the most exciting thing ever, and I'm not going to lie to you and say it will be easy to develop your expression-simplifying skills, but the payoff is totally worth it.

In Chapter 3 we'll discuss exponents and logarithms in detail.

For instance, the expression a^x means to multiply a by itself x times. Exponents have some interesting properties, which we'll spend some time exploring. We'll also discuss the logarithmic function $\log_a(x)$, which is the inverse function of the exponential function a^x .

In Chapter 4 we'll discuss the Cartesian plane, which allows us to visualize math concepts like points, vectors, and geometric shapes. We can also use the Cartesian plane to plot function graphs, which are visual representations of how functions behave for all inputs.

Chapter 5 is dedicated to the study of functions. We'll introduce important new vocabulary for describing functions' inputs, outputs, and other properties. We'll build a catalogue of the 10 most important functions for math and scientific modelling. We'll also discuss how to combine, modify, and transform basic functions in order to form composite functions with various desirable properties.

The subject of geometry deals with shapes, transformations, distances, areas, and volumes. In Chapter 6 we'll learn about geometry, and discuss various formulas for lines, circles, cylinders, spheres, and other shapes. We'll also discuss trigonometry—the study of the relative lengths of the sides of right-angle triangles.

In Chapter 7 we'll discuss vectors, which are used to describe directions in space. Vectors are useful for science calculations, computing, statistics, machine learning, and myriad other domains.

In Chapter 8 we'll discuss various supplementary math topics. We'll see how to solve systems of linear equations, how to compute compound interest, and learn a bit about proofs and set notation. Learning set notation (Section 8.3) is a key foundation for studying more advanced math topics.

The book concludes with a sizeable set of problems (Chapter 9). Take note, dear reader: math is not a spectator sport. Reading math is not enough—you must also practice your skills by solving problems.

Use the maps

The concept maps on pages iii and iv show an overview of the concepts and topics we'll discuss in the book. Note the numerous connections between concepts. Understanding these connections is the key to understanding math: math isn't a random bunch of facts, but an interconnected network of ideas.

Are you ready for this? Let's get started!

Chapter 1

Numbers, variables, and equations

In this chapter we'll introduce the fundamental building blocks of math: numbers, variables, equations, and functions. Learning about the basics will establish a solid foundation for all other topics that are to come.

We'll start by learning a general procedure for solving equations. This is a fundamental skill you'll develop throughout the book, starting in Section 1.1 and expanding as you learn about more advanced types of equations and solution techniques. In Section 1.2 we'll define the different types of numbers that exist, then discuss number representations in Section 1.3. In Section 1.4 we'll learn how to use variables like k and x to represent unknowns in equations.

We'll then discuss functions and function inverses in Section 1.5. A function describes a transformation between an input variable x and a corresponding output variable y . The inverse function describes the same transformation in the opposite direction, treating y as the input and x as the output. By learning how to apply functions and function inverses to solve equations, you're taking one giant step toward acquiring math superpowers. Overall, your knowledge of numbers, variables, equations, and functions will allow you to solve many types of problems on the journey ahead.

1.1 Solving equations

Most math skills boil down to being able to manipulate and solve equations. Solving an equation means finding the value of the unknown in the equation.

Check this shit out:

$$x^2 - 4 = 45.$$

To solve the above equation is to answer the question “What is x ?” More precisely, we want to find the number that can take the place of x in the equation so that the equality holds. In other words, we’re asking,

“Which number times itself minus four gives 45?”

That is quite a mouthful, don’t you think? To remedy this verbosity, mathematicians often use specialized symbols to describe math operations. The problem is that these specialized symbols can be very confusing. Sometimes even the simplest math concepts are inaccessible if you don’t know what the symbols mean.

What are your feelings about math, dear reader? Are you afraid of it? Do you have anxiety attacks because you think it will be too difficult for you? Chill! Relax, my brothers and sisters. There’s nothing to it. Nobody can magically guess the solution to an equation immediately. To find the solution, you must break the problem into simpler steps. Let’s walk through this one together.

To find x , we can manipulate the original equation, transforming it into a different equation (as true as the first) that looks like this:

$$x = \text{only numbers.}$$

That’s what it means to *solve* an equation: the equation is solved because the unknown is isolated on one side, while the constants are grouped on the other side. You can type the numbers on the right-hand side into a calculator and obtain the numerical value of x .

By the way, before we continue our discussion, let it be noted: the equality symbol ($=$) means that all that is to the left of $=$ is equal to all that is to the right of $=$. To keep this equality statement true, **for every change you apply to the left side of the equation, you must apply the same change to the right side of the equation.**

To find x , we need to manipulate the original equation into its final form, simplifying it step by step until it can’t be simplified any further. The only requirement is that the manipulations we make transform one true equation into another true equation. In this example, the first simplifying step is to add the number four to both sides of the equation:

$$x^2 - 4 + 4 = 45 + 4,$$

which simplifies to

$$x^2 = 49.$$

Now the expression looks simpler, yes? How did I know to perform this operation? I wanted to “undo” the effects of the operation -4 . We undo an operation by applying its *inverse*. In the case where the operation is the subtraction of some amount, the inverse operation is the addition of the same amount. We’ll learn more about function inverses in Section 1.5.

We’re getting closer to our goal of *isolating* x on one side of the equation, leaving only numbers on the other side. The next step is to undo the square x^2 operation. The inverse operation of squaring a number x^2 is to take its square root $\sqrt{\quad}$, so that’s what we’ll do next. We obtain

$$\sqrt{x^2} = \sqrt{49}.$$

Notice how we applied the square root to both sides of the equation? If we don’t apply the same operation to both sides, we’ll break the equality!

The equation $\sqrt{x^2} = \sqrt{49}$ simplifies to

$$|x| = 7.$$

What’s up with the vertical bars around x ? The notation $|x|$ stands for the *absolute value* of x , which is the same as x except we ignore the sign that indicates whether x is positive or negative. For example $|5| = 5$ and $|-5| = 5$, too. The equation $|x| = 7$ indicates that both $x = 7$ and $x = -7$ satisfy the equation $x^2 = 49$. Seven squared is 49, $7^2 = 49$, and negative seven squared is also 49, $(-7)^2 = 49$, because the two negative signs cancel each other out.

The final solutions to the equation $x^2 - 4 = 45$ are

$$x = 7 \quad \text{and} \quad x = -7.$$

Yes, there are *two* possible answers. You can check that both of the above values of x satisfy the initial equation $x^2 - 4 = 45$.

If you are comfortable with all the notions of high school math and you feel you could have solved the equation $x^2 - 4 = 45$ on your own, then you can skim through this chapter quickly. If on the other hand you are wondering how the squiggle killed the power two, then this chapter is for you! In the following sections we will review all the essential concepts from high school math that you will need to power through the rest of this book. First, let me tell you about the different kinds of numbers.

1.2 Numbers

In the beginning, we must define the main players in the world of math: numbers.

Definitions

Numbers are the basic objects we use to count, measure, quantify, and calculate things. Mathematicians like to classify the different kinds of number-like objects into categories called *sets*:

- The natural numbers: $\mathbb{N} = \{0, 1, 2, 3, 4, 5, 6, 7, \dots\}$
- The integers: $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$
- The rational numbers: $\mathbb{Q} = \{\frac{5}{3}, \frac{22}{7}, 1.5, 0.125, -7, \dots\}$
- The real numbers: $\mathbb{R} = \{-1, 0, 1, \sqrt{2}, e, \pi, 4.94\dots, \dots\}$
- The complex numbers: $\mathbb{C} = \{-1, 0, 1, i, 1 + i, 2 + 3i, \dots\}$

These categories of numbers should be somewhat familiar to you. Think of them as neat classification labels for everything that you would normally call a number. Each group in the above list is a *set*. A set is a collection of items of the same kind. Each collection has a name and a precise definition for which items belong in that collection. Note also that each of the sets in the list contains all the sets above it, as illustrated in Figure 1.1. For now, we don't need to go into the details of sets and set notation, but we do need to be aware of the different sets of numbers.

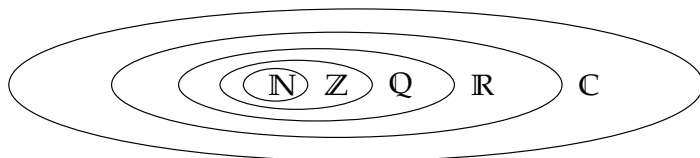


Figure 1.1: An illustration of the nested containment structure of the different number sets. The set of natural numbers is contained in the set of integers, which in turn is contained in the set of rational numbers. The set of rational numbers is contained in the set of real numbers, which is contained in the set of complex numbers.

Why do we need so many different sets of numbers? Each set of numbers is associated with more and more advanced mathematical problems.

The simplest numbers are the natural numbers \mathbb{N} , which are sufficient for all your math needs if all you're going to do is *count* things. How many goats? Five goats here and six goats there so the total is

number 2 corresponds to a line segment of length two, which is a geometric object in a one-dimensional space. If we add a line segment of length two in a second dimension, we obtain a square with area 2^2 in a two-dimensional space. Adding a third dimension, we obtain a cube with volume 2^3 in a three-dimensional space. Indeed, raising a base a to the exponent 2 is commonly called “ a squared,” and raising a to the power of 3 is called “ a cubed.”

The geometrical analogy about one-dimensional quantities as lengths, two-dimensional quantities as areas, and three-dimensional quantities as volumes is good to keep in mind.

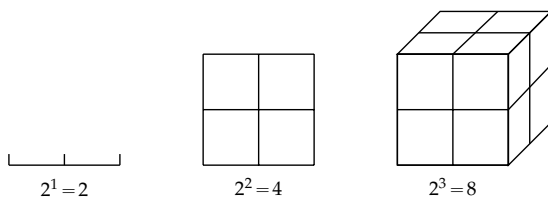


Figure 1.4: Geometric interpretation for exponents 1, 2, and 3. A length raised to exponent 2 corresponds to the area of a square. The same length raised to exponent 3 corresponds to the volume of a cube.

Our visual intuition works very well up to three dimensions, but we can use other means of visualizing higher exponents, as demonstrated in Figure 1.5.

Operator precedence

There is a standard convention for the order in which mathematical operations must be performed. The basic algebra operations have the following precedence:

1. Parentheses
2. Exponents
3. Multiplication and Division
4. Addition and Subtraction

If you’re seeing this list for the first time, the acronym PEMDAS and the associated mnemonic “Please Excuse My Dear Aunt Sally,” might help you remember the order of operations.

For instance, the expression $5 \cdot 3^2 + 13$ is interpreted as “First find the square of 3, then multiply it by 5, and then add 13.” Parentheses are needed to carry out the operations in a different order: to multiply 5 times 3 first and *then* take the square, the equation should read

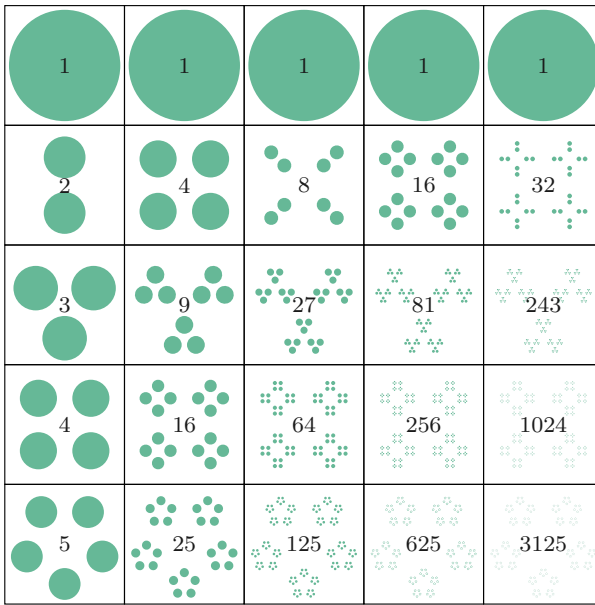


Figure 1.5: Visualization of numbers raised to different exponents. Each box in this grid contains a^n dots, where the base a varies from one through five, and the exponent n varies from one through five. In the first row we see that the number $a = 1$ raised to any exponent is equal to itself. The second row corresponds to the base $a = 2$ so the number of dots doubles each time we increase the exponent by one. Starting from $2^1 = 2$ in the first column, we end up with $2^5 = 32$ in the last column. The rest of the rows show how exponentiation works for different bases.

$(5 \cdot 3)^2 + 13$, where parentheses indicate that the square acts on $(5 \cdot 3)$ as a whole and not on 3 alone.

Exercises

E1.1 Solve for the unknown x in the following equations:

a) $3x + 2 - 5 = 4 + 2$

b) $\frac{1}{2}x - 3 = \sqrt{3} + 12 - \sqrt{3}$

c) $\frac{7x-4}{2} + 1 = 8 - 2$

d) $5x - 2 + 3 = 3x - 5$

E1.2 Indicate all the number sets the following numbers belong to.

a) -2

b) $\sqrt{-3}$

c) $8 \div 4$

d) $\frac{5}{3}$

e) $\frac{\pi}{2}$

E1.3 Calculate the values of the following expressions:

a) $2^3 3 - 3$

b) $2^3(3 - 3)$

c) $\frac{4-2}{3^5}(6 \cdot 7 - 41)$

voila, you can rewrite the equation in terms of the variable u :

$$\frac{6}{5-u} = u,$$

which contains no square roots.

The next step to solve for u is to undo the division operation. Multiply both sides of the equation by $(5-u)$ to obtain

$$\frac{6}{5-u}(5-u) = u(5-u),$$

which simplifies to

$$6 = 5u - u^2.$$

This can be rewritten as the equation $u^2 - 5u + 6 = 0$, which in turn can be rewritten as $(u-2)(u-3) = 0$ using the techniques we'll learn in Section 2.2.

We now see that the solutions are $u_1 = 2$ and $u_2 = 3$. The last step is to convert our u -answers into x -answers by using $u = \sqrt{x}$, which is equivalent to $x = u^2$. The final answers are $x_1 = 2^2 = 4$ and $x_2 = 3^2 = 9$. Try plugging these x values into the original square root equation to verify that they satisfy it.

Compact notation

Symbolic manipulation is a powerful tool because it allows us to manage complexity. Say you're solving a physics problem in which you're told the mass of an object is $m = 140$ kg. If there are many steps in the calculation, would you rather use the number 140 kg in each step, or the shorter symbol m ? It's much easier to use m throughout your calculation, and wait until the last step to substitute the value 140 kg when computing the final numerical answer.

1.5 Functions and their inverses

As we saw in the section on solving equations, the ability to “undo” functions is a key skill for solving equations.

Example Suppose we're solving for x in the equation

$$f(x) = c,$$

where f is some function and c is some constant. We're looking for the unknown x such that $f(x)$ equals c . Our goal is to isolate x on one side of the equation, but the function f stands in our way.

By using the *inverse function* (denoted f^{-1}) we “undo” the effects of f . We apply the inverse function f^{-1} to both sides of the equation to obtain

$$f^{-1}(f(x)) = f^{-1}(c).$$

By definition, the inverse function f^{-1} performs the opposite action of the function f , so together the two functions cancel each other out. We have $f^{-1}(f(x)) = x$ for any number x .

Provided everything is kosher (the function f^{-1} must be defined for the input c), the manipulation we made above is valid and we have obtained the answer $x = f^{-1}(c)$.

The above example introduces the notation f^{-1} for denoting the inverse function. This notation is inspired by the notation for reciprocals. Recall that multiplication by the reciprocal number a^{-1} is the inverse operation of multiplication by the number a : $a^{-1}ax = 1x = x$. In the case of functions, however, the negative-one exponent does not refer to “one over- $f(x)$ ” as in $\frac{1}{f(x)} = (f(x))^{-1}$; rather, it refers to the inverse function. In other words, the number $f^{-1}(y)$ is equal to the number x such that $f(x) = y$.

Be careful: sometimes an equation can have multiple solutions. For example, the function $f(x) = x^2$ maps two input values (x and $-x$) to the same output value $x^2 = f(x) = f(-x)$. The inverse function of $f(x) = x^2$ is $f^{-1}(y) = \sqrt{y}$, but both $x = +\sqrt{c}$ and $x = -\sqrt{c}$ are solutions to the equation $x^2 = c$. In this case, this equation’s solutions can be indicated in shorthand notation as $x = \pm\sqrt{c}$.

Formulas

Here is a list of common functions and their inverses:

function $f(x)$	\Leftrightarrow	inverse $f^{-1}(x)$
$x + 2$	\Leftrightarrow	$x - 2$
$2x$	\Leftrightarrow	$\frac{1}{2}x$
$-1x$	\Leftrightarrow	$-1x$
x^2	\Leftrightarrow	$\pm\sqrt{x}$
2^x	\Leftrightarrow	$\log_2(x)$
$3x + 5$	\Leftrightarrow	$\frac{1}{3}(x - 5)$
a^x	\Leftrightarrow	$\log_a(x)$
$\exp(x) = e^x$	\Leftrightarrow	$\ln(x) = \log_e(x)$
$\sin(x)$	\Leftrightarrow	$\sin^{-1}(x) = \arcsin(x)$

Chapter 2

Algebra

The rules of algebra tell us how to manipulate math expressions. By observing and working with the structure of expressions, we can simplify them, and thus compute their values more easily. In this chapter we'll learn the **general rules that apply to all math expressions**. Knowing these rules will equip you with a powerful toolbox of tricks you can apply to any problem you might face.

In Section 1.1, we learned how to *isolate* the unknown x in equations, which is by far the most important technique for solving equations. In this chapter we'll learn a few more techniques, including how to *simplify*, *expand*, and *factor* expressions. Together, these techniques are called *algebra*, from the Arabic *al-jabr* meaning “reunion of broken parts.”

In Section 2.2, we'll use algebra to derive a general formula for solving quadratic equations of the form $ax^2 + bx + c = 0$. Knowing how to solve quadratic equations is really important because they show up all over the place.

Alright! Are you ready for algebra? Let's get started!

2.1 Basic rules of algebra

It's important that you know the general rules for manipulating numbers and variables, a process otherwise known as—you guessed it—*algebra*. This little refresher will cover these concepts to make sure you're comfortable on the algebra front. We'll also review some important algebraic tricks, like *factoring* and *completing the square*, which are useful when solving equations.

Let's define some terminology for referring to different parts of math expressions. When an expression contains multiple things added together, we call those things *terms*. Furthermore, terms are

usually composed of many things multiplied together. When a number x is obtained as the product of other numbers like $x = abc$, we say “ x factors into a , b , and c .” We call a , b , and c the *factors* of x .

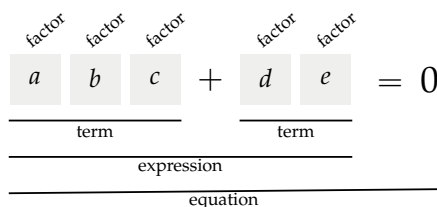


Figure 2.1: Diagram showing the names used to describe the different parts of the equation $abc + de = 0$.

Given any three numbers a , b , and c , we can apply the following algebraic properties:

1. Associative property: $a + b + c = (a + b) + c = a + (b + c)$ and $abc = (ab)c = a(bc)$
2. Commutative property: $a + b = b + a$ and $ab = ba$
3. Distributive property: $a(b + c) = ab + ac$

We use the distributive property every time we *expand* brackets. For example $a(b + c + d) = ab + ac + ad$. The brackets, also known as parentheses, indicate the expression $(b + c + d)$ must be treated as a whole; as a factor consisting of three terms. Multiplying this expression by a is the same as multiplying each term by a .

The opposite operation of expanding is called *factoring*, which consists of rewriting the expression with the common parts taken out in front of a bracket: $ab + ac = a(b + c)$. In this section, we'll discuss all algebra operations and illustrate what they're capable of.

Example Suppose we are asked to solve for t in the equation

$$7(3 + 4t) = 11(6t - 4).$$

Since the unknown t appears on both sides of the equation, it is not immediately obvious how to proceed.

To solve for t , we can bring all t terms to one side and all constant terms to the other side. First, expand the two brackets to obtain

$$21 + 28t = 66t - 44.$$

Then move things around to relocate all t s to the equation's right-hand side and all constants to the left-hand side:

$$21 + 44 = 66t - 28t.$$

There's no linear term because the $-xp$ term cancels the $+px$ term. Any time you see an expression like $x^2 - p^2$, you can know it comes from a product of the form $(x + p)(x - p)$.

A *perfect square* is a quadratic expression that can be written as the product of repeated factors $(x + p)$:

$$x^2 + 2px + p^2 = (x + p)(x + p) = (x + p)^2.$$

Note $x^2 - 2qx + q^2 = (x - q)^2$ is also a perfect square.

Completing the square

In this section we'll learn about an ancient algebra technique called *completing the square*, which allows us to rewrite *any* quadratic expression of the form $x^2 + Bx + C$ as a perfect square plus some constant correction factor $(x + p)^2 + k$. This algebra technique was described in one of the first books on *al-jabr* (algebra), written by Al-Khwarizmi around the year 800 CE. The name "completing the square" comes from the ingenious geometric construction used by this procedure. Yes, we can use geometry to solve algebra problems!

We assume the starting point for the procedure is a quadratic expression whose quadratic coefficient is one, $x^2 + Bx + C$, and use capital letters B and C to denote the linear and constant coefficients. The capital letters are to avoid any confusion with the quadratic expression $ax^2 + bx + c$, for which $a \neq 1$. Note we can always write $ax^2 + bx + c$ as $a(x^2 + \frac{b}{a}x + \frac{c}{a})$ and apply the procedure to the expression inside the brackets, identifying $\frac{b}{a}$ with B and $\frac{c}{a}$ with C .

First let's rewrite the quadratic expression $x^2 + Bx + C$ by splitting the linear term into two equal parts:

$$x^2 + \frac{B}{2}x + \frac{B}{2}x + C.$$

We can interpret the first three terms geometrically as follows: the x^2 term corresponds to a square with side length x , while the two $\frac{B}{2}x$ terms correspond to rectangles with sides $\frac{B}{2}$ and x . See the left side of Figure 2.2 for an illustration.

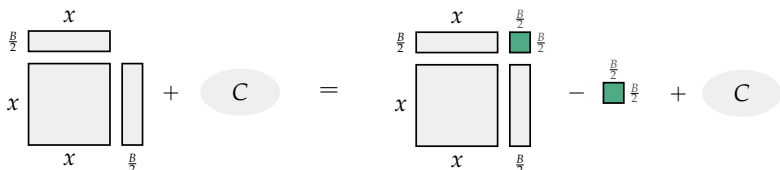


Figure 2.2: To complete the square in the expression $x^2 + Bx + C$, we need to add the quantity $(\frac{B}{2})^2$, which corresponds to a square (shown in darker colour) with sides equal to half the coefficient of the linear term. We also subtract $(\frac{B}{2})^2$ so the overall value of the expression remains unchanged.

The square with area x^2 and the two rectangles can be positioned to form a larger square with side length $(x + \frac{B}{2})$. Note there's a small piece of sides $\frac{B}{2}$ by $\frac{B}{2}$ missing from the corner. To *complete the square*, we can add a term $(\frac{B}{2})^2$ to this expression. To preserve the equality, we also subtract $(\frac{B}{2})^2$ from the expression to obtain:

$$\begin{aligned} x^2 + \frac{B}{2}x + \frac{B}{2}x + C &= \underbrace{x^2 + \frac{B}{2}x + \frac{B}{2}x + (\frac{B}{2})^2}_{(x + \frac{B}{2})^2} - (\frac{B}{2})^2 + C \\ &= (x + \frac{B}{2})^2 - (\frac{B}{2})^2 + C. \end{aligned}$$

The right-hand side of this equation describes the area of the square with side length $(x + \frac{B}{2})$, minus the area of the small square $(\frac{B}{2})^2$, plus the constant C , as illustrated on the right side of Figure 2.2.

We can summarize the entire procedure in one equation:

$$x^2 + Bx + C = \underbrace{(x + \frac{B}{2})^2}_{(1)} + \underbrace{C - (\frac{B}{2})^2}_{(2)}.$$

There are two things to remember when you want to apply the complete-the-square trick: (1) choose the constant inside the bracket to be $\frac{B}{2}$ (half of the linear coefficient), and (2) subtract $(\frac{B}{2})^2$ outside the bracket in order to keep the equation balanced.

Solving quadratic equations

Suppose we want to solve the quadratic equation $x^2 + Bx + C = 0$. It's not possible to solve this equation with the digging-toward-the- x approach from Section 1.1 (since x appears in both the quadratic term x^2 and the linear term Bx). Enter the completing-the-square trick!

Example Let's find the solutions of the equation $x^2 + 5x + 6 = 0$. The coefficient of the linear term is $B = 5$, so we choose $\frac{B}{2} = \frac{5}{2}$ for the constant inside the bracket, and subtract $(\frac{B}{2})^2 = (\frac{5}{2})^2$ outside the bracket to keep the equation balanced. Completing the square gives

$$x^2 + 5x + 6 = (x + \frac{5}{2})^2 + 6 - (\frac{5}{2})^2 = 0.$$

Next we use fraction arithmetic to simplify the constant terms in the expression: $6 - (\frac{5}{2})^2 = 6 \cdot \frac{4}{4} - \frac{25}{4} = \frac{24-25}{4} = \frac{-1}{4} = -0.25$.

We're left with the equation

$$(x + 2.5)^2 - 0.25 = 0,$$

Quadratic formula

The solutions to the equation $ax^2 + bx + c = 0$ for $a \neq 0$ are

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

The quadratic formula is usually abbreviated $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, where the sign “ \pm ” stands for both “ $+$ ” and “ $-$.” The notation “ \pm ” allows us to express both solutions x_1 and x_2 in one equation, but you should keep in mind there are really two solutions.

Let’s see how the quadratic formula is used to solve the equation $2x^2 - 4x - 6 = 0$. Finding the two solutions requires the simple mechanical task of identifying $a = 2$, $b = -4$, and $c = -6$, then plugging these values into the two parts of the formula:

$$\begin{aligned} x_1 &= \frac{4 + \sqrt{4^2 - 4(2)(-6)}}{4} = \frac{4 + \sqrt{16 + 48}}{4} = \frac{4 + \sqrt{64}}{4} = 3, \\ x_2 &= \frac{4 - \sqrt{4^2 - 4(2)(-6)}}{4} = \frac{4 - \sqrt{16 + 48}}{4} = \frac{4 - \sqrt{64}}{4} = -1. \end{aligned}$$

We can easily verify that value $x_1 = 3$ and $x_2 = -1$ both satisfy the original equation $2x^2 = 4x + 6$.

Proof of the quadratic formula

Every claim made by a mathematician comes with a *proof*, which is a step-by-step argument that shows why the claim is true. It’s easy to see where a proof starts and where a proof ends in mathematical texts. Each proof begins with the heading *Proof* (usually in italics) and has the symbol “ \square ” at its end. The purpose of these demarcations is to give readers the option to skip the proof. It’s not necessary to read and understand the proofs of all math statements, but reading proofs can often lead you to a more solid understanding of the material.

I want you to see the proof of the quadratic formula because it’s an important result that you’ll use very often to solve math problems. Reading the proof will help you understand where the quadratic formula comes from. The proof relies on the completing-the-square technique from the previous section, and some basic algebra operations. You can totally handle this!

Chapter 3

Exponents and logarithms

Understanding exponents and logarithms is important for your math modelling skills. Exponential functions can describe population growth, interest calculations on loans, probabilities of random events, voltages in electric capacitors, radioactive decay, and many other phenomena.

The logarithmic function is the inverse of the exponential function. We need the logarithmic function every time we want to solve an equation containing exponents. Additionally, the logarithmic scale allows us to conveniently compare numbers of vastly different sizes. Logarithmic scales are used to describe systems with exponential scale differences, including sound intensity and the acidity of chemical solutions.

As a student of math, you can consider exponentiation and logarithm computation as basic operations, on the same footing as addition, subtraction, multiplication, and division.

Basic math operations: $\{ x + y, x - y, xy, x/y, x^y, \log_x y \}$.

In this chapter, you'll learn the basic rules for manipulating exponents and logarithms. As you read and practice, your job is to make sure you're just as comfortable with exponents as you are with addition and multiplication, and as comfortable with logarithms as you are with subtraction and division.

3.1 Exponents

In math we must often multiply together the same number many times, so we use the notation

$$b^n = \underbrace{bbb \cdots bb}_{n \text{ times}}$$

to denote some number b multiplied by itself n times. In this section we'll review the basic terminology associated with exponents and discuss their properties.

Definitions

The fundamental ideas of exponents are:

- b^n : the number b raised to the power n
 - ▷ b : the *base*
 - ▷ n : the *exponent* or *power* of b in the expression b^n

By definition, the zeroth power of any number is equal to one, expressed as $b^0 = 1$.

We'll also discuss *exponential functions*. In particular, we define the following important exponential functions:

- b^x : the exponential function base b
- 10^x : the exponential function base 10
- $\exp(x) = e^x$: the exponential function base e . The number e is called *Euler's number*.
- 2^x : the exponential function base 2. This function is important in computer science.

The number $e = 2.7182818\ldots$ is a special base with many applications. We call e the *natural* base. Another special base is 10 because we use the decimal system for our numbers. We can write very large numbers and very small numbers as powers of 10. For example, one thousand can be written as $1\,000 = 10^3$, one million is $1\,000\,000 = 10^6$, and one billion is $1\,000\,000\,000 = 10^9$.

Formulas

The following properties follow from the definition of exponentiation as repeated multiplication.

E3.2 Simplify the following expressions as much as possible:

a) $\sqrt{3}\sqrt{3}$

b) $\sqrt{9}\sqrt{16}$

c) $\frac{\sqrt[3]{8}\sqrt{3}}{\sqrt{4}}$

d) $\frac{\sqrt{aba}}{\sqrt{b}}$

E3.3 Calculate the values of the following exponential expressions:

a) $\sqrt{2}(\pi)2^{\frac{1}{2}}$

b) $8^{\frac{2}{3}} + 8^{-\frac{2}{3}}$

c) $\left(\frac{(\sqrt[3]{c})^3}{c}\right)^{77}$

d) $\left(\frac{x^2\sqrt{x^4}}{x^3}\right)^2$

E3.4 Find all the values of x that satisfy these equations:

a) $x^2 = a$

b) $x^3 = b$

c) $x^4 = c$

d) $x^5 = d$

E3.5 Coulomb's constant k_e is defined by the formula $k_e = \frac{1}{4\pi\epsilon_0}$, where ϵ_0 is the permittivity of free space. Use a calculator to compute the value of k_e starting from $\epsilon_0 = 8.854 \times 10^{-12}$ and $\pi = 3.14159265$. Report your answer with an appropriate number of digits, even if the calculator gives you a number with more digits.

3.2 Logarithms

Some people think the word “logarithm” refers to some mythical, mathematical beast. Legend has it that logarithms are many-headed, breathe fire, and are extremely difficult to understand. Nonsense! Logarithms are simple. It will take you at most a couple of pages to get used to manipulating them, and that is a good thing because logarithms are used all over the place.

The strength of your sound system is measured in logarithmic units called decibels. This is because your ears are sensitive only to exponential differences in sound intensity. Logarithms allow us to compare very large numbers and very small numbers on the same scale. If sound were measured in linear units instead of logarithmic units, your sound system's volume control would need to range from 1 to 1 048 576. That would be weird, no? This is why we use the logarithmic scale for volume notches. Using a logarithmic scale, we can go from sound intensity level 1 to sound intensity level 1 048 576 in 20 “progressive” steps. Assume each notch doubles the sound intensity, rather than increasing the intensity by a fixed amount. If the first notch corresponds to 2, the second notch is 4—still probably inaudible, turn it up! By the time you get to the sixth notch you're at $2^6 = 64$ sound intensity, which is the level of audible music. The tenth notch corresponds to sound intensity $2^{10} = 1024$ (medium-strength sound), and finally the twentieth notch reaches a max power of $2^{20} = 1\,048\,576$, at which point the neighbours come to complain.

Chapter 4

Coordinate systems

Numbers have a dual nature: they're simultaneously abstract mathematical concepts and concrete real-world quantities. The number four corresponds to the abstract concept of "fourness" that is shared by all groups of four objects, but four can also be represented as the length of a four-metre-long wooden pole. This dual nature of numbers is what makes them so useful: we can use numbers and algebra to solve real-world problems, and also visualize abstract algebra problems in terms of real-world lengths. If you want to know the combined length of a four-metre pole and a two-metre pole, you can perform the calculation using numbers inside your head, without needing to move any real poles. Alternatively, you can learn about the abstract concepts of addition and multiplication by manipulating physical objects.¹

It is possible to add more "dimensions" to the correspondence between numbers and lengths. The Cartesian plane is a two-dimensional *coordinate system* that assigns a pair of coordinates (x, y) to every point in a two-dimensional plane. In this chapter we'll discuss this important visualization tool that will support your understanding of functions (Chapter 5), geometry (Chapter 6), and vectors (Chapter 7).

Many important math ideas can be expressed visually as two-dimensional pictures, geometric shapes, and curves. The same math ideas can also be described by abstract math concepts like numbers and equations. Coordinate systems serve as the bridge between the space you can visualize and the space of abstract math concepts like coordinates and equations.

¹For example, playing with Cuisenaire rods is an excellent way to become familiar with numbers and math operations.

4.1 The Cartesian plane

The Cartesian plane, named after famous philosopher and mathematician René Descartes, is used to visualize pairs of numbers (x, y) .

Recall the number line representation for numbers that we introduced in Section 1.3.

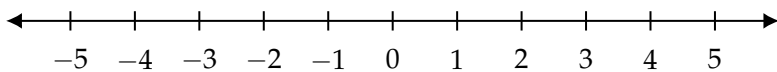


Figure 4.1: Every real number x corresponds to a point on the number line. The number line extends indefinitely to the left (toward negative infinity) and to the right (toward positive infinity).

The Cartesian plane is the two-dimensional generalization of the number line. Generally, we call the plane's horizontal axis “the x -axis” and its vertical axis “the y -axis.” We put notches at regular intervals on each axis so we can measure distances.

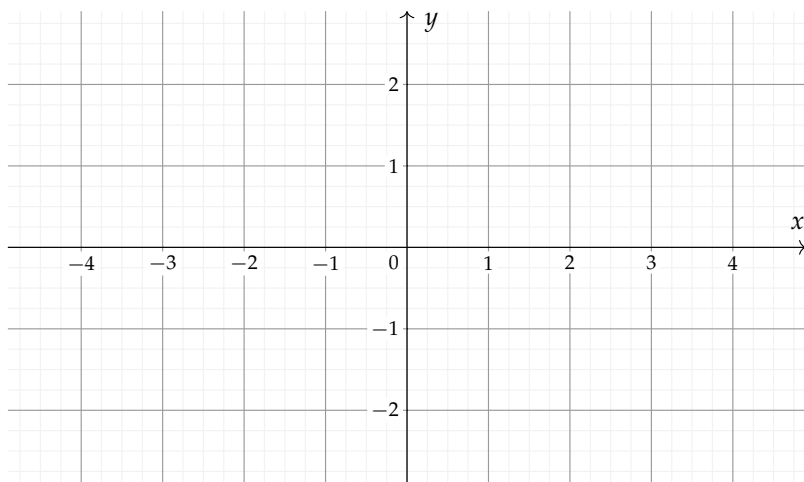


Figure 4.2: Every point in the Cartesian plane corresponds to a pair of real numbers (x, y) . Points $P = (P_x, P_y)$, vectors $\vec{v} = (v_x, v_y)$, and graphs of functions $(x, f(x))$ live here.

Figure 4.2 is an example of an empty Cartesian coordinate system. Think of the coordinate system as an empty canvas. What can you draw on this canvas?

Chapter 5

Functions

Learning math is like learning a new language. One of the most useful ideas you can learn in the language of math is the concept of a function. If numbers and variables correspond to nouns, functions correspond to verbs. Specifically, functions codify relations between variables; any relationship between two variables can be expressed as a function. This makes functions a powerful tool for modelling real-world situations.

In this chapter we'll learn some important math verbs. We'll start with some general theory about functions, then catalogue the most important functions that occur in everyday life, in scientific modelling, and in business. We're building your vocabulary of math action verbs so you'll be able to get in on the action.

In Section 5.1 we'll introduce the main concepts used to characterize functions: the function's domain and image, the function's graph, and the function's values. Section 5.2 serves as a reference manual for 10 functions of central importance in math, science, and engineering. In Section 5.3 we'll observe how translation and scaling transformations affect the functions' graphs.

5.1 Functions

We need to have a relationship talk. We need to talk about functions. We use functions to describe the relationships between variables. In particular, functions describe how one variable *depends* on another.

For example, the revenue R from a music concert depends on the number of tickets sold n . If each ticket costs \$25, the revenue from the concert can be written *as a function of n* as follows: $R(n) = 25n$. Solving for n in the equation $R(n) = 7000$ tells us the number of ticket sales needed to generate \$7000 in revenue. This is a simple

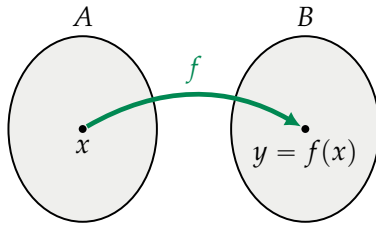


Figure 5.1: An abstract representation of a function f from the set A to the set B . The function f is the arrow which *maps* each input x in A to an output $f(x)$ in B . The output of the function $f(x)$ is also denoted y .

- B : the *target set* of a function describes the type of outputs the function has. The target set is sometimes called the *codomain*.
- $\text{Im}(f)$: the *image* of the function is the set of all possible output values of the function. The image is sometimes called the *range*.

See Figure 5.2 for an illustration of these concepts. The purpose of introducing all this math terminology is so we'll have words to distinguish the general types of inputs and outputs of the function (real numbers, complex numbers, vectors) from the specific properties of the function like its domain and image.

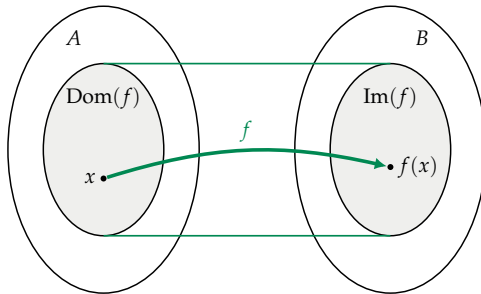


Figure 5.2: Illustration of the input and output sets of a function $f: A \rightarrow B$. The *source set* is denoted A and the *domain* is denoted $\text{Dom}(f)$. Note that the function's domain is a subset of its source set. The *target set* is denoted B and the *image* is denoted $\text{Im}(f)$. The image is a subset of the target set.

Let's look at an example to illustrate the difference between the source set and the domain of a function. Consider the square root function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined as $f(x) = \sqrt{x}$, which is shown in Figure 5.3. The source set of f is the set of real numbers—yet only non-negative real numbers are allowed as inputs, since \sqrt{x} is not defined for negative numbers. Therefore, the domain of the square root function is only the nonnegative real numbers: $\text{Dom}(f) = \mathbb{R}_+ = \{x \in$

equation $f(x) = b$ has *at least one* solution for every number b . If the function f is bijective then it is both injective and surjective, which means the equation $f(x) = b$ has *exactly one* solution.

Inverse function

We used inverse functions repeatedly in previous chapters, each time describing the inverse function informally as an “undo” operation. Now that we have learned about bijective functions, we can give a the precise definition of the inverse function and explain some of the details we glossed over previously.

Recall that a *bijective* function $f : A \rightarrow B$ is a *one-to-one correspondence* between the numbers in the source set A and numbers in the target set B : for every output y , there is exactly one corresponding input value x such that $f(x) = y$. The *inverse function*, denoted f^{-1} , is the function that takes any output value y in the set B and finds the corresponding input value x that produced it $f^{-1}(y) = x$.

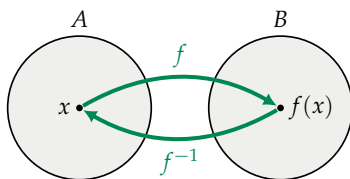


Figure 5.6: The inverse f^{-1} undoes the operation of the function f .

For every bijective function $f : A \rightarrow B$, there exists an inverse function $f^{-1} : B \rightarrow A$ that performs the *inverse mapping* of f . If we start from some x , apply f , and then apply f^{-1} , we’ll arrive—full circle—back to the original input x :

$$f^{-1}(f(x)) = x.$$

In Figure 5.6 the function f is represented as a forward arrow, and the inverse function f^{-1} is represented as a backward arrow that puts the value $f(x)$ back to the x it came from.

Similarly, we can start from any y in the set B and apply f^{-1} followed by f to get back to the original y we started from:

$$f(f^{-1}(y)) = y.$$

In words, this equation tells us that f is the “undo” operation for the function f^{-1} , the same way f^{-1} is the “undo” operation for f .

If a function is missing the injective property or the surjective property then it isn’t bijective and it doesn’t have an inverse. Without

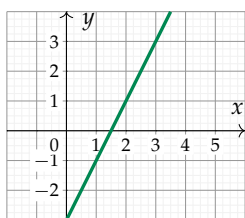
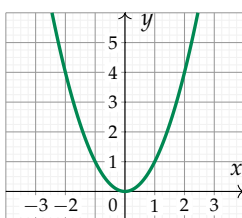
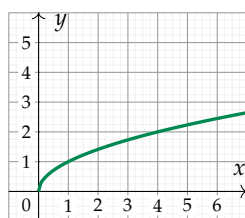
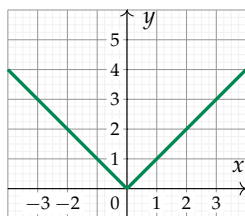
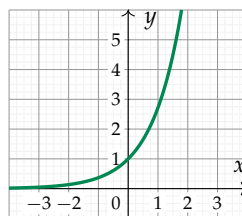
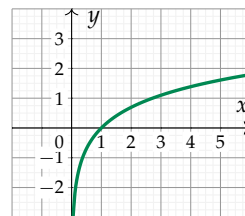
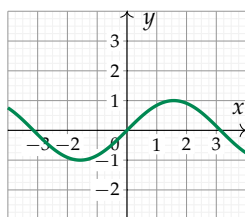
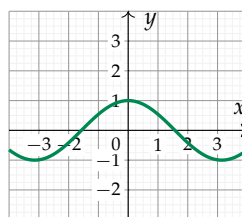
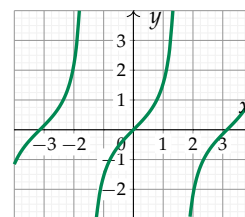
(a) $f(x) = 2x - 3$ (b) $f(x) = x^2$ (c) $f(x) = \sqrt{x}$ (d) $f(x) = |x|$ (e) $f(x) = e^x$ (f) $f(x) = \ln(x)$ (g) $f(x) = \sin(x)$ (h) $f(x) = \cos(x)$ (i) $f(x) = \tan(x)$

Figure 5.9: We'll see many types of function graphs in the next pages.

Line

The equation of a line describes an input-output relationship where the change in the output is *proportional* to the change in the input. The equation of a line is

$$f(x) = mx + b.$$

The constant m describes the slope of the line. The constant b is called the y -intercept and it is the value of the function when $x = 0$.

Consider what relationship the equation of $f(x)$ describes for different values of m and b . What happens when m is positive? What happens when m is negative?

Square root

The square root function is denoted

$$f(x) = \sqrt{x} = x^{\frac{1}{2}}.$$

The square root \sqrt{x} is the inverse function of the square function x^2 when the two functions are defined as $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$. The symbol \sqrt{c} refers to the *positive* solution of $x^2 = c$. Note that $-\sqrt{c}$ is also a solution of $x^2 = c$.

Graph

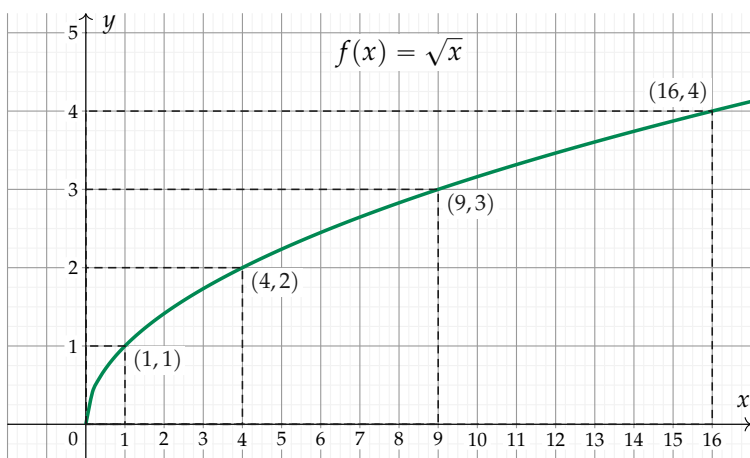


Figure 5.12: The graph of the function $f(x) = \sqrt{x}$. The domain of the function is \mathbb{R}_+ because we can't take the square root of a negative number.

Properties

- **Domain:** $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$. The function $f(x) = \sqrt{x}$ is only defined for nonnegative inputs. There is no real number y such that y^2 is negative, hence the function $f(x) = \sqrt{x}$ is not defined for negative inputs x .
- **Image:** $\mathbb{R}_+ = \{y \in \mathbb{R} \mid y \geq 0\}$. The outputs of the function $f(x) = \sqrt{x}$ are nonnegative numbers since $\sqrt{x} \geq 0$.

In addition to *square* root, there is also *cube* root $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$, which is the inverse function for the cubic function $f(x) = x^3$. We have $\sqrt[3]{8} = 2$ since $2 \times 2 \times 2 = 8$. More generally, we can define the n^{th} -root function $\sqrt[n]{x}$ as the inverse function of x^n .

Natural logarithm

The natural logarithm function is denoted

$$f(x) = \ln(x) = \log_e(x).$$

The function $\ln(x)$ is the inverse function of the exponential e^x .

Graph

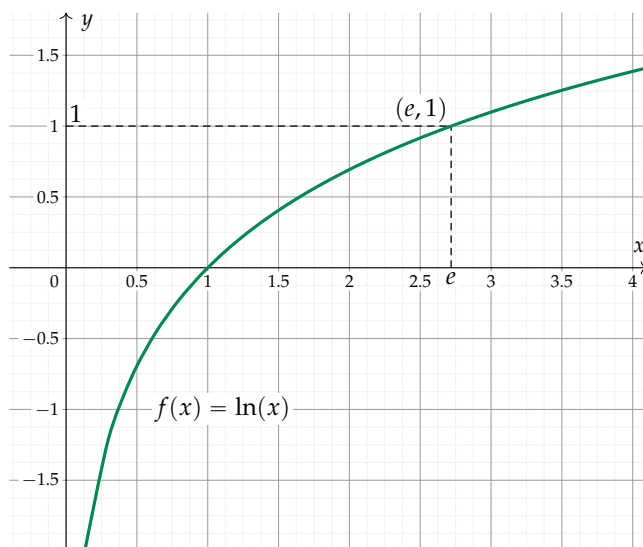


Figure 5.20: The graph of the function $\ln(x)$ passes through the following coordinates: $(\frac{1}{e^2}, -2)$, $(\frac{1}{e}, -1)$, $(1, 0)$, $(e, 1)$, $(e^2, 2)$, $(e^3, 3)$, $(e^4, 4)$, etc.

Properties

- Domain: $\{x \in \mathbb{R} \mid x > 0\}$
- Image: \mathbb{R}

Exercises

E5.1 Find the domain, the image, and the roots of $f(x) = 2 \cos(x)$.

E5.2 What are the degrees of the following polynomials? Are they even, odd, or neither?

a) $p(x) = x^2 - 5x^4 + 1$

b) $q(x) = x - x^3 + x^5 - x^7$

E5.3 Solve for x in the following polynomial equations.

a) $3x + x^2 = x - 15 + 2x^2$

b) $3x^2 - 4x - 4 + x^3 = x^3 + 2x + 2$

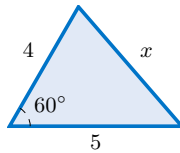
Chapter 6

Geometry

Geometry is the mathematical study of shapes and proportions. This branch of mathematics encompasses all things that can be drawn and analyzed using geometric concepts like similarity, proportionality, and orthogonality. Geometry studies the common, abstract nature of all geometrical shapes. For example, instead of studying and describing the properties of any particular triangle, we'll study the geometrical properties common to all triangles.

We won't just look at shapes; we'll develop a language for describing them quantitatively in terms of lengths and angles. We can find patterns in different geometric shapes and describe proportions and lengths using equations.

Here's the chapter breakdown. In Section 6.1, we'll discuss properties of several geometric shapes: triangles, circles, cylinders, spheres, cones, and pyramids. Sections 6.2 and 6.3 discuss the topic of *trigonometry*, which is the study of the proportions of the sides of right-angle triangles. We'll learn all about the trigonometric functions $\sin\theta$ and $\cos\theta$, which play an important role in many areas of mathematics. In Section 6.4 we'll learn more about circles and *polar coordinates*. Sections 6.5, 6.6, and 6.7 will discuss the *ellipse*, the *parabola*, and the *hyperbola*, which are three other interesting geometric shapes that occur in nature.



Hint: Use the cosine rule.

E6.2 Find the volume and the surface area of a sphere with radius 2.

E6.3 On a rainy day, Laura brings her bike indoors, and the wet bicycle tires leave a track of water on the floor. What is the length of the water track left by the bike's rear tire (diameter 73 cm) if the wheel makes five full turns along the floor?

6.2 Trigonometry

If one of the angles in a triangle is equal to 90° , we call this triangle a *right-angle triangle*. In this section we'll discuss right-angle triangles in great detail and get to know their properties. We'll learn some fancy new terms like *hypotenuse*, *opposite*, and *adjacent*, which are used to refer to the different sides of a triangle. We'll also use the functions *sine*, *cosine*, and *tangent* to compute the *ratios of lengths* in right triangles.

Understanding triangles and their associated trigonometric functions is of fundamental importance: you'll need this knowledge for your future understanding of mathematical concepts like vectors and complex numbers.

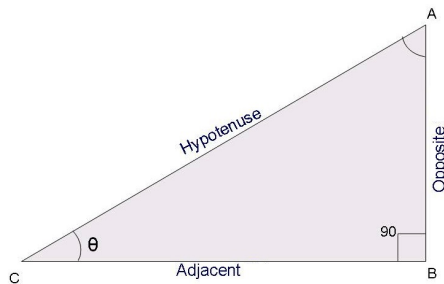


Figure 6.8: A right-angle triangle. The angle at the base is denoted θ and the names of the sides of the triangle are indicated.

Concepts

- A, B, C : the three *vertices* of the triangle
- θ : the angle at the vertex C . Angles can be measured in degrees or radians.

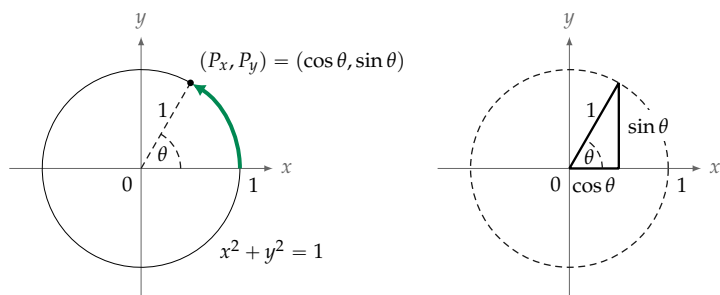


Figure 6.9: The unit circle corresponds to the equation $x^2 + y^2 = 1$. The coordinates of the point P on the unit circle are $P_x = \cos \theta$ and $P_y = \sin \theta$.

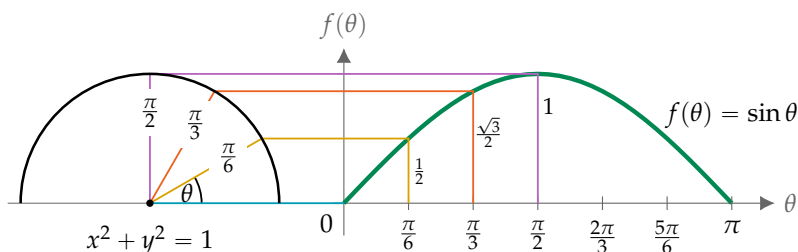


Figure 6.10: The function $f(\theta) = \sin \theta$ describes the vertical position of a point P that travels along the unit circle. The graph shows the values of the function $f(\theta) = \sin \theta$ for angles between $\theta = 0$ and $\theta = \pi$.

are the most common angles that often show up on homework and exam questions. For each angle, the x -coordinate (the first number in the bracket) is $\cos \theta$, and the y -coordinate (the second number in the bracket) is $\sin \theta$.

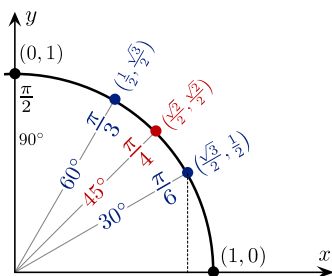


Figure 6.11: The combined $(\cos \theta, \sin \theta)$ coordinates for the points on the unit circle at the most common angles: 0 , $\frac{\pi}{6}$ (30°), $\frac{\pi}{4}$ (45°), $\frac{\pi}{3}$ (60°), and $\frac{\pi}{2}$ (90°).

Note the values of $\cos \theta$ and $\sin \theta$ for the angles shown in Figure 6.11 are all combinations of the fractions $\frac{1}{2}$, $\frac{\sqrt{2}}{2}$, and $\frac{\sqrt{3}}{2}$. The square

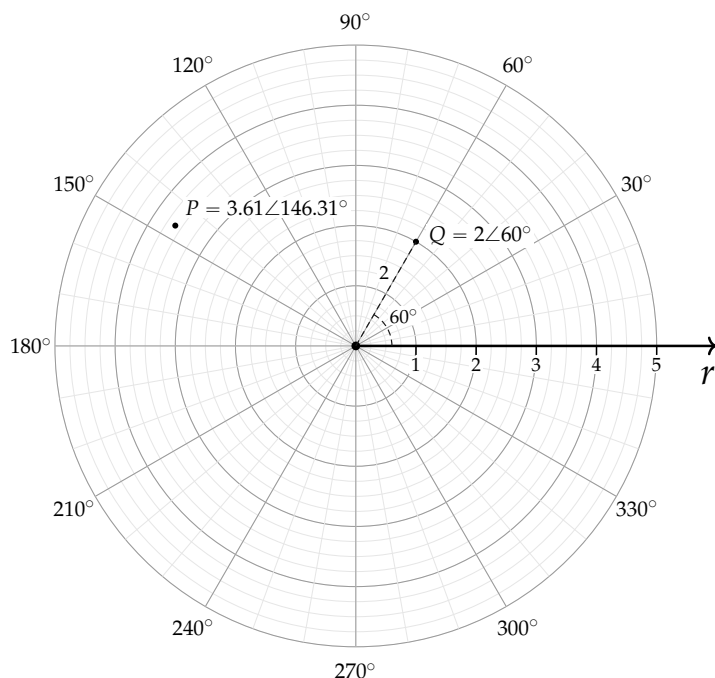


Figure 6.15: We can use the polar coordinate system to describe points in the two-dimensional plane. The polar coordinates $r \angle \theta$ describe the point located at the distance r from the origin in the direction θ .

A Cartesian coordinate pair (x, y) is made of x and y coordinates, while a polar coordinate pair $r \angle \theta$ is made of r and θ coordinates. In this book, we use the angle symbol \angle (read “at an angle of”) to separate the polar coordinates r and θ , in order to emphasize the difference between Cartesian and polar coordinates. However, some other books use the notation (r, θ) for polar coordinates, so you have to watch out—the coordinate pair $(20, 30)$ could be either a (x, y) coordinate pair or a (r, θ) coordinate pair, depending on the context.

Note the polar coordinates that describe a given point are not unique, meaning the same point can be described in multiple ways. The point $Q = 2 \angle 60^\circ$ is equally described by the polar coordinates $2 \angle -300^\circ$, since a clockwise turn of 300° is the same as a counter-clockwise turn of 60° . We can also describe the same point Q using the polar coordinates $-2 \angle 240^\circ$ and $-2 \angle -120^\circ$, which tell us to turn in the direction opposite to 60° and measure a negative distance $r = -2$. While all of these polar coordinates for Q are equivalent, the preferred way to specify polar coordinates is with positive r values and angles $|\theta| \leq 180^\circ$.

- b : the semi-minor axis of the ellipse is the half-width of the ellipse along the y -axis. The distance between V_3 and V_4 is $2b$.
- c : the distance of the focal points from the centre of the ellipse. The distance between F_1 and F_2 is $2c$.
- ϵ : the *eccentricity* of the ellipse, $\epsilon = \sqrt{1 - \frac{b^2}{a^2}} = \frac{c}{a}$.

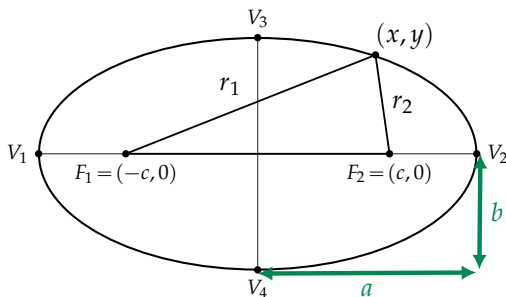


Figure 6.20: An ellipse with semi-major axis a and semi-minor axis b . The locations of the focal points F_1 and F_2 are indicated.

Definition

An ellipse is the curve found by tracing along all the points for which the sum of the distances to the two focal points is a constant:

$$r_1 + r_2 = \text{const.}$$

There's a neat way to draw a perfect ellipse using a piece of string and two tacks. Take a piece of string and tack it to a picnic table at two points, leaving some loose slack in the middle of the string. Now take a pencil, and without touching the table, use the pencil to pull the middle of the string until it is taut. Make a mark at that point. With the two parts of string completely straight, make a mark at every point possible where the two "legs" of string remain taut.

An ellipse is a set of points (x, y) that satisfy the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The parameters a and b determine the shape of the ellipse.

The focal points F_1 and F_2 correspond to the locations of the two tacks where the string is held in place. The coordinates of the two focal points are

$$F_1 = (-c, 0) \quad \text{and} \quad F_2 = (c, 0),$$

where $c = \sqrt{a^2 - b^2}$ is the *focal distance*.

The *eccentricity* of an ellipse is given by the equation

$$\varepsilon = \sqrt{1 - \frac{b^2}{a^2}} = \frac{c}{a}.$$

The parameter ε (the Greek letter *epsilon*) varies between 0 and 1 and describes how much the shape of the ellipse differs from the shape of a circle. When $\varepsilon = 0$ the ellipse is a circle with radius a , and both focal points are located at the centre. As the eccentricity ε increases, the ellipse becomes more elongated and the focal points spread farther apart.

Polar coordinates

Consider a polar coordination system whose centre is located at the focus F_2 . We can describe the ellipse by specifying the function $r_2(\theta)$, which describes the distance from the focus F_2 to the point E on the ellipse as a function of the angle θ (see Figure 6.21). Recall that for functions in polar coordinates, the angle θ is the independent variable that varies from 0 to 2π (360°), and the dependent variable is the distance $r_2(\theta)$.

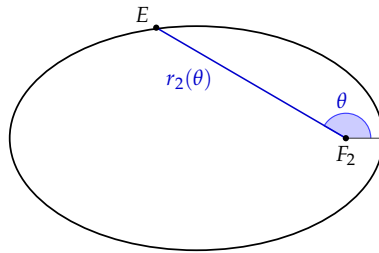


Figure 6.21: The function $r_2(\theta)$ in polar coordinates specifies the distance between the point E on the ellipse and the focal point F_2 for all angles.

The function that describes an ellipse in polar coordinates is

$$r_2(\theta) = \frac{a(1 - \varepsilon^2)}{1 + \varepsilon \cos(\theta)},$$

where the angle θ is measured with respect to the semi-major axis. The distance is smallest when $\theta = 0$ with $r_2(0) = a - c = a(1 - \varepsilon)$ and largest when $\theta = \pi$ with $r_2(\pi) = a + c = a(1 + \varepsilon)$.

Calculating the orbit of the Earth

The motion of the Earth around the Sun is an ellipse with the Sun positioned at the focus F_2 . We can therefore use the polar coordinates

formula $r_2(\theta)$ to describe the distance of the Earth from the Sun. The eccentricity of Earth's orbit around the Sun is $\varepsilon = 0.01671123$, and the half-length of the major axis is $a = 149\,598\,261$ km. We substitute these values into the general formula for $r_2(\theta)$ and obtain the following equation:

$$r_2(\theta) = \frac{149\,556\,484}{1 + 0.01671123 \cos(\theta)} \text{ km.}$$

The point where the Earth is closest to the Sun is called the *perihelion*. It occurs when $\theta = 0$, which happens around the 3rd of January. The moment where the Earth is most distant from the Sun is called the *aphelion* and corresponds to the angle $\theta = \pi$. Earth's *aphelion* happens around the 3rd of July.

Let's use the formula for $r_2(\theta)$ to predict the *perihelion* and *aphelion* distances of Earth's orbit:

$$r_{2,\text{peri}} = r_2(0) = \frac{149556483}{1 + 0.01671123 \cos(0)} = 147\,098\,290 \text{ km,}$$

$$r_{2,\text{aphe}} = r_2(\pi) = \frac{149556483}{1 + 0.01671123 \cos(\pi)} = 152\,098\,232 \text{ km.}$$

Google "perihelion" and "aphelion" to verify that the above predictions are accurate. It's kind of cool that a mathematical formula can describe the motion of our planet, don't you think?

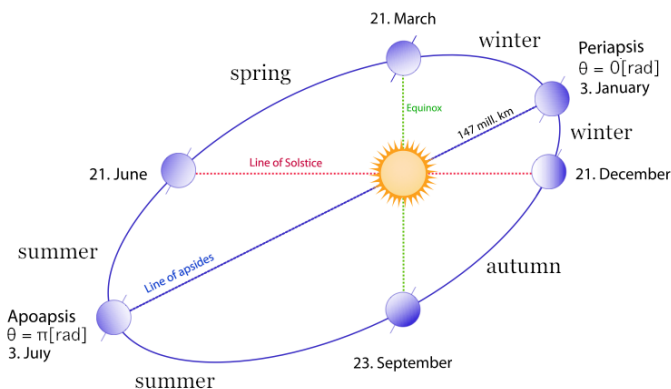


Figure 6.22: The orbit of the Earth around the Sun. Key points of the orbit are labelled. The seasons in the Northern hemisphere are also indicated.

The angle θ of the Earth relative to the Sun can be described as a function of time $\theta(t)$. The exact formula of the function $\theta(t)$ that describes the angle as a function of time is fairly complicated, so we

6.6 Parabola

The *parabola* is another important geometric shape. In this section, we'll see how we can describe parabolas using their geometric properties, as well as in terms of algebraic equations.

Parameters

Figure 6.23 shows a parabola with all its parameters annotated:

- f : the *focal length* of the parabola
- $F = (0, f)$: the *focal point* of the parabola
- $\{(x, y) \in \mathbb{R}^2 \mid y = -f\}$: the *directrix* line to the parabola
- r : the distance from point P on the parabola to the focal point F
- ℓ : the closest distance from a point P on the parabola to the parabola's directrix line

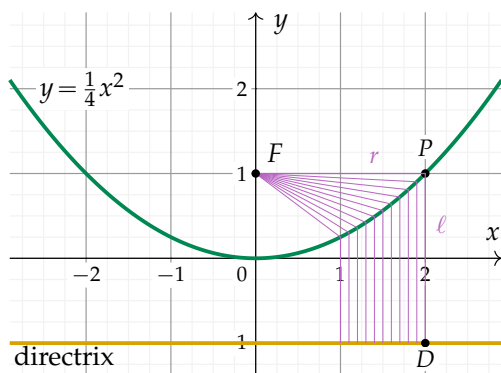


Figure 6.23: The parabola is defined geometrically as the set of points whose distance from the focal point r is equal to their distance from the directrix ℓ . The figure shows the point P on the parabola that has distance $r = 2$ from F , and distance $\ell = 2$ from the point D on the directrix. This parabola can be described algebraically using the equation $y = \frac{1}{4}x^2$.

Geometric definition

The shape of a parabola is determined by a single parameter f , called the *focal length*. For a parabola with focal length f , the focal point is at $F = (0, f)$ and the *directrix* line has the equation $y = -f$. The parabola is defined as the set of points P for which the distance from the focal point and the directrix are equal:

$$r = \ell,$$

6.7 Hyperbola

The *hyperbola* is another fundamental shape of nature.

Parameters

- F_1, F_2 : the *focal points* of the hyperbola
- r_1 : the distance from a point of the hyperbola to F_1
- r_2 : the distance from a point of the hyperbola to F_2
- a : the semi-major axis of the hyperbola is the distance from the origin to the vertices V_1 and V_2
- b : the semi-minor axis of the hyperbola is the distance from a focus to the nearest asymptote
- c : the distance of the focal points from the centre. The distance between F_1 and F_2 is $2c$.
- ε : *eccentricity* of the hyperbola, $\varepsilon = \sqrt{1 + \frac{b^2}{a^2}} = \frac{c}{a}$

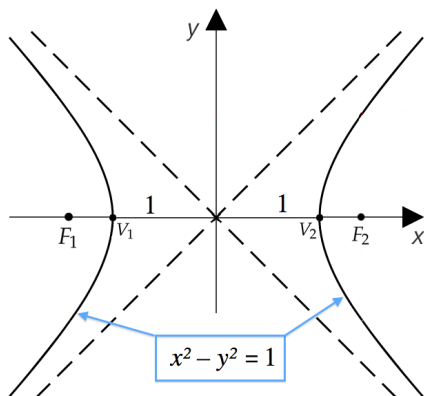


Figure 6.26: The graph of the unit hyperbola $x^2 - y^2 = 1$. The graph has two branches opening to the sides, and its eccentricity is $\varepsilon = \sqrt{1 + \frac{1}{1}} = \sqrt{2}$.

The graph of a hyperbola consists of two separate *branches*, as illustrated in Figure 6.26. The dashed lines are called the *asymptotes* of the hyperbola. The graph of the hyperbola approaches these lines but never touches them. The equations that describe these asymptotes are $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$.

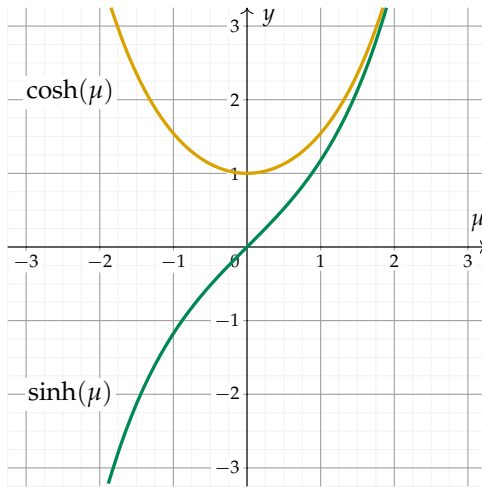


Figure 6.28: The graphs of the functions $\cosh \mu$ and $\sinh \mu$.

Don't worry about $\cosh \mu$ and $\sinh \mu$ too much. The hyperbolic trig functions are used much less often than the circular trigonometric functions $\cos \theta$ and $\sin \theta$. The main thing to remember is the general pattern: cosine functions are used to denote horizontal coordinates and sine functions are used to denote vertical coordinates.

The conic sections

There is a deep connection between the geometric shapes of the circle, the ellipse, the parabola, and the hyperbola. These seemingly different shapes can be obtained, geometrically speaking, from a single object: the cone. We can obtain the four curves by slicing the cone at different angles, as illustrated in Figure 6.29.

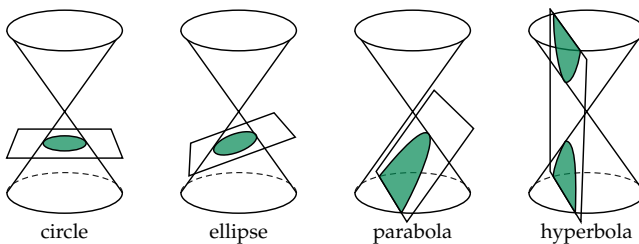


Figure 6.29: Taking slices through a cone at different angles produces different geometric shapes: a circle, an ellipse, a parabola, or a hyperbola.

Conic sections in polar coordinates

All four conic sections can be described by the same function in polar coordinates:

$$r(\theta) = \frac{q(1 + \varepsilon)}{1 + \varepsilon \cos(\theta)},$$

where q is the curve's closest distance to a focal point and ε is the curve's eccentricity. For a circle, $q = R$ (the radius) and the eccentricity parameter is $\varepsilon = 0$. For an ellipse, $q = a(1 - \varepsilon)$ and the eccentricity parameter varies between 0 and 1 ($0 \leq \varepsilon < 1$). Note we include the case $\varepsilon = 0$ since a circle is a special case of an ellipse. For a parabola, $q = f$ (the focal length) and the eccentricity is $\varepsilon = 1$. For a hyperbola, $q = a(\varepsilon - 1)$ and the eccentricity is $\varepsilon > 1$.

We can use the eccentricity parameter ε to classify all four curves. Depending on the value of ε , the equation $r(\theta)$ defines either a circle, an ellipse, a parabola, or a hyperbola. Table 6.4 summarizes all our observations regarding conic sections.

Conic section	Equation	Polar function	Eccentricity
Circle	$x^2 + y^2 = R^2$	$r(\theta) = R$	$\varepsilon = 0$
Ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$r(\theta) = \frac{a(1-\varepsilon^2)}{1+\varepsilon \cos(\theta)}$	$\varepsilon = \sqrt{1 - \frac{b^2}{a^2}}, 0 \leq \varepsilon < 1$
Parabola	$y^2 = 4fx$	$r(\theta) = \frac{2f}{1+\cos(\theta)}$	$\varepsilon = 1$
Hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$r(\theta) = \frac{a(\varepsilon^2-1)}{1+\varepsilon \cos(\theta)}$	$\varepsilon = \sqrt{1 + \frac{b^2}{a^2}}, 1 < \varepsilon < \infty$

Table 6.4: The four conic sections and their eccentricity parameters.

The motion of the planets is explained by Newton's law of gravitation. The gravitational interaction between two bodies always leads one of the two bodies to follow a trajectory described by one of the conic sections for which the other body is the focal point. Figure 6.30 illustrates four different trajectories for a satellite near planet F . The circle ($\varepsilon = 0$) and the ellipse ($0 \leq \varepsilon < 1$) describe *closed orbits*, in which the satellite is captured in the gravitational field of the planet F and remains in orbit forever. The parabola ($\varepsilon = 1$) and the hyperbola ($\varepsilon > 1$) describe *open orbits*, in which the satellite swings by the planet F and then continues.

Links

[Interactive graph of a hyperbola]

<https://www.desmos.com/calculator/2mnsk5o8vn>

Chapter 7

Vectors

In this chapter we'll learn how to manipulate multi-dimensional objects called vectors. Vectors are the precise way to describe directions in space. We need vectors in order to describe physical quantities like forces, velocities, and accelerations.

Vectors are built from ordinary numbers, which form the *components* of the vector. You can think of a vector as a list of numbers, and *vector algebra* as operations performed on the numbers in the list. Vectors can also be manipulated as geometric objects, represented by arrows in space. For instance, the arrow that corresponds to the vector $\vec{v} = (v_x, v_y)$ starts at the origin $(0, 0)$ and ends at the point (v_x, v_y) . The word vector comes from the Latin *vehere*, which means *to carry*. Indeed, the vector \vec{v} takes the point $(0, 0)$ and carries it to the point (v_x, v_y) .

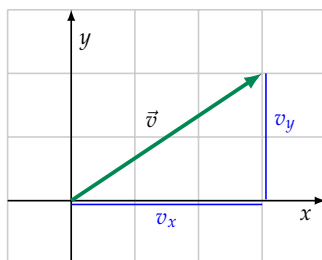


Figure 7.1: The vector $\vec{v} = (3, 2)$ is an arrow in the Cartesian plane. The horizontal component of \vec{v} is $v_x = 3$ and the vertical component is $v_y = 2$.

This chapter will introduce you to vectors, vector algebra, and vector operations, which are very useful for solving physics problems. What you'll learn here applies more broadly to current problems in computer graphics, probability theory, machine learning, and other

fields of science and mathematics. It's all about vectors these days, so you'd best get to know them.

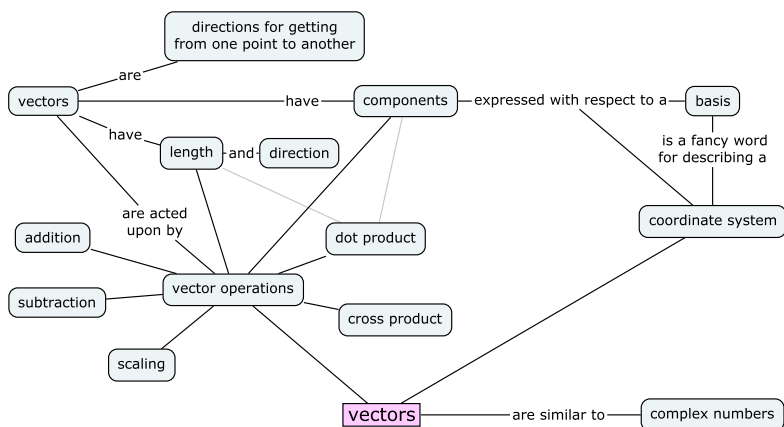


Figure 7.2: This figure illustrates the new concepts related to vectors. As you can see, there is quite a bit of new vocabulary to learn, but don't be fazed—all these terms are just fancy ways of talking about arrows.

7.1 Great outdoors

Vectors are directions for getting from point A to point B. Directions can be given in terms of street names and visual landmarks, or with respect to a coordinate system.

While on vacation in British Columbia, you want to visit a certain outdoor location your friend told you about. Your friend isn't available to take you there himself, but he has sent you *directions* for how to get to the place from the bus stop:

Sup G. Go to bus stop number 345. Bring a compass.
Walk 2 km north then 3 km east. You will find X there.

This text message contains all the information you need to find X.

Act 1: Following directions

You arrive at the bus stop, which is located at the top of a hill. From this height you can see the whole valley, and along the hillside below spreads a beautiful field of tall crops. The crops are so tall they prevent anyone standing in them from seeing too far; good thing you have a compass. You align the compass needle so the red arrow points north. You walk 2 km north, then turn 90° to the right

Vector as arrows

So far, we described how to perform algebraic operations on vectors in terms of their components. Vector operations can also be interpreted geometrically, as operations on arrows in the Cartesian plane.

Vector addition The sum of two vectors corresponds to the combined displacement of the two vectors. Figure 7.4 illustrates the addition of two vectors, $\vec{v}_1 = (3, 0)$ and $\vec{v}_2 = (2, 2)$. The sum of the two vectors is the vector $\vec{v}_1 + \vec{v}_2 = (3, 0) + (2, 2) = (5, 2)$.

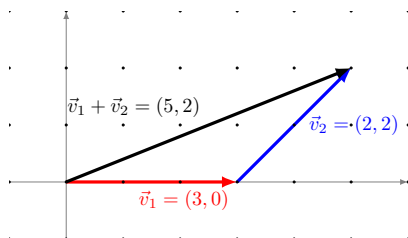


Figure 7.4: The addition of the vectors \vec{v}_1 and \vec{v}_2 produces the vector $(5, 2)$.

Vector subtraction Before we describe vector subtraction, note that multiplying a vector by a scale factor $\alpha = -1$ gives a vector of the same length as the original, but pointing in the opposite direction.

This fact is useful if you want to subtract two vectors using the graphical approach. Subtracting a vector is the same as adding the negative of the vector:

$$\vec{w} - \vec{v}_1 = \vec{w} + (-\vec{v}_1) = \vec{v}_2.$$

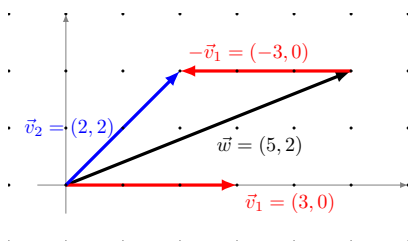


Figure 7.5: The vector subtraction $\vec{w} - \vec{v}_1$ is equivalent to the vector addition $\vec{w} + (-\vec{v}_1)$, where $(-\vec{v}_1)$ is like \vec{v}_1 but points in the opposite direction.

Figure 7.5 illustrates the graphical procedure for subtracting the vector $\vec{v}_1 = (3, 0)$ from the vector $\vec{w} = (5, 2)$. Subtraction of $\vec{v}_1 = (3, 0)$ is the same as addition of $-\vec{v}_1 = (-3, 0)$.

Scaling The scaling operation acts to change the length of a vector. Suppose we want to obtain a vector in the same direction as the vector $\vec{v} = (3, 2)$, but half as long. “Half as long” corresponds to a scale factor of $\alpha = 0.5$. The scaled-down vector is $\vec{w} = 0.5\vec{v} = (1.5, 1)$. Conversely, we can think of the vector \vec{v} as being twice as long as the vector \vec{w} .

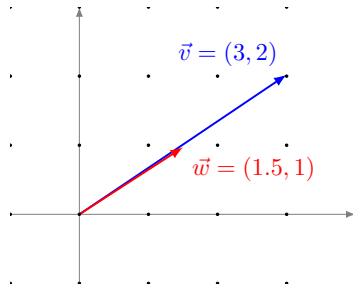


Figure 7.6: Vectors \vec{v} and \vec{w} are related by the equation $\vec{v} = 2\vec{w}$.

Multiplying a vector by a negative number reverses its direction.

Length-and-direction representation

So far, we’ve seen how to represent a vector in terms of its components. There is another way of representing two-dimensional vectors: we can describe the vector $\vec{v} \in \mathbb{R}^2$ in terms of its length $\|\vec{v}\|$ and its direction θ —the angle it makes with the x -axis. For example, the vector $(1, 1)$ can also be written as $\sqrt{2}/45^\circ$ in polar coordinates. This length-and-direction notation is useful because it makes it easy to see the “size” of vectors. On the other hand, vector arithmetic operations are much easier to carry out in the component notation. It’s therefore good to know the formulas for converting between the two vector representations.

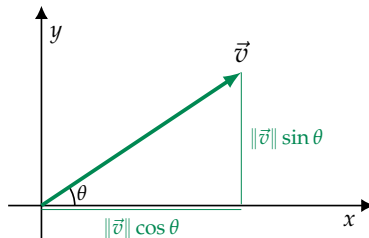


Figure 7.7: The x - and y -components of a vector with length $\|\vec{v}\|$ in the direction θ are given by $v_x = \|\vec{v}\| \cos \theta$ and $v_y = \|\vec{v}\| \sin \theta$.

To convert the length-and-direction vector $\vec{v} = \|\vec{v}\| \angle \theta$ into an x -component and a y -component (v_x, v_y) , use the formulas

$$v_x = \|\vec{v}\| \cos \theta \quad \text{and} \quad v_y = \|\vec{v}\| \sin \theta.$$

To convert from component notation $\vec{v} = (v_x, v_y)$ to length-and-direction $\|\vec{v}\| \angle \theta$, use

$$\|\vec{v}\| = \sqrt{v_x^2 + v_y^2}, \quad \theta = \begin{cases} \tan^{-1}\left(\frac{v_y}{v_x}\right) & \text{if } v_x > 0, \\ 180^\circ + \tan^{-1}\left(\frac{v_y}{v_x}\right) & \text{if } v_x < 0, \\ 90^\circ & \text{if } v_x = 0 \text{ and } v_y > 0, \\ -90^\circ & \text{if } v_x = 0 \text{ and } v_y < 0. \end{cases}$$

We've already seen these formulas in Section 6.4 (page 111), when we learned about the transformations between Cartesian and polar coordinates for points. The conversion procedure for vectors is exactly the same, including the trickiness around calculating θ when v_x is negative or zero. I invite you to revisit exercise E6.8 on page 115 to review the conversion operations between Cartesian coordinates and polar coordinates.

Unit vector notation

In two dimensions, we can think of a vector $\vec{v} = (v_x, v_y)$ as a command to "Go a distance v_x in the x -direction and a distance v_y in the y -direction." To write this set of commands more explicitly, we can use multiples of the vectors \hat{i} and \hat{j} . These are the unit vectors pointing in the x and y directions:

$$\hat{i} = (1, 0) \quad \text{and} \quad \hat{j} = (0, 1).$$

Any number multiplied by \hat{i} corresponds to a vector with that number in the first coordinate. For example, $3\hat{i} = (3, 0)$ and $4\hat{j} = (0, 4)$.

In physics, we tend to perform a lot of numerical calculations with vectors; to make things easier, we often use unit vector notation:

$$v_x \hat{i} + v_y \hat{j} \quad \Leftrightarrow \quad (v_x, v_y).$$

The addition rule remains the same for the new notation:

$$\underbrace{2\hat{i} + 3\hat{j}}_{\vec{v}} + \underbrace{5\hat{i} - 2\hat{j}}_{\vec{w}} = \underbrace{7\hat{i} + 1\hat{j}}_{\vec{v} + \vec{w}}.$$

It's the same story repeating all over again: we need to add \hat{i} s with \hat{i} s, and \hat{j} s with \hat{j} s.

learning we often represent “rich data” like images, videos, and text as vectors with thousands of dimensions.

An example of an n -dimensional vector is

$$\vec{v} = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n.$$

The vector algebra operations you learned in this section also apply to these high-dimensional vectors.

Vectors and vector coordinates

One final point we need to clarify is the difference between real-world vector quantities like the velocity of a tennis ball \vec{v} and its mathematical representation as a coordinate vector (v_x, v_y, v_z) . If you know the coordinate vector (v_x, v_y, v_z) then you know what the real-world velocity is, right? Not quite.

Let’s say you’re doing a physics research project on tennis serves. You define an xyz -coordinate system for the tennis court, which allows you to represent the ball’s velocity \vec{v} as a triple of components (v_x, v_y, v_z) interpreted as: “The ball is moving with velocity v_x units in the x -direction, v_y units in the y -direction, and v_z units in the z -direction.”

Suppose you want to describe the velocity vector \vec{v} to a fellow physicist via text message. Referring to your sheet of calculations, you find the values $\vec{v} = (60, 3, -2)$, which you know were measured in metres per second. You send this message:

The velocity is (60,3,-2) measured in metres per second.

A few minutes later the following reply comes back:

Wait whaaat? What coordinate system are you using?

Indeed the information you sent is incomplete. Vector components depend on the coordinate system in which the vectors are represented. The triple of numbers $(60, 3, -2)$ only makes sense once you know the directions of the axes in the xyz -coordinate system. Realizing your mistake, you send a text with all the required information:

Using the coordinate system centred at the south post of the net, with the x -axis pointing east along the court, the y -axis pointing north along the net, and the z -axis pointing up, the velocity is (60,3,-2) in metres per second.

A few seconds later, you get the reply:

OK got it now. Thx!

This hypothetical situation illustrates the importance of the coordinate systems for describing vectors. If you don't know what the coordinate system is, knowing the coordinates (v_x, v_y, v_z) doesn't tell you much. Only when you know the directions of the unit vectors \hat{i} , \hat{j} , and \hat{k} can you interpret the instructions $\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$.

It turns out, using the *xyz*-coordinate system with the three vectors $\{\hat{i}, \hat{j}, \hat{k}\}$ is just one of many possible ways we can represent vectors. We can represent a vector \vec{v} as coordinates (v_1, v_2, v_3) with respect to any *basis* $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ using the expression $\vec{v} = v_1\hat{e}_1 + v_2\hat{e}_2 + v_3\hat{e}_3$, which corresponds to the instructions: "Move v_1 units in the direction of \hat{e}_1 , move v_2 units in the direction of \hat{e}_2 , and move v_3 units in the direction of \hat{e}_3 ."

What's a basis, you ask? I'm glad you asked, because this is the subject of the next section.

7.3 Basis

One of the most important concepts in the study of vectors is the concept of a *basis*. Consider the three-dimensional vector space \mathbb{R}^3 . A *basis* for \mathbb{R}^3 is a set of vectors $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ that can be used as a coordinate system for \mathbb{R}^3 . If the set of vectors $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ is a basis, then you can *represent* any vector $\vec{v} \in \mathbb{R}^3$ as coordinates (v_1, v_2, v_3) *with respect to* that basis:

$$\vec{v} = v_1\hat{e}_1 + v_2\hat{e}_2 + v_3\hat{e}_3.$$

The vector \vec{v} is obtained by measuring out a distance v_1 in the \hat{e}_1 direction, a distance v_2 in the \hat{e}_2 direction, and a distance v_3 in the \hat{e}_3 direction.

You are already familiar with the *standard* basis $\{\hat{i}, \hat{j}, \hat{k}\}$, which is associated with the *xyz*-coordinate system. You know that any vector $\vec{v} \in \mathbb{R}^3$ can be expressed as a triple (v_x, v_y, v_z) with respect to the basis $\{\hat{i}, \hat{j}, \hat{k}\}$ through the formula $\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$. The whole point of this section is to let you know that other bases (coordinate systems) exist, and to get you into the habit of asking, "With respect to which coordinate system?" every time you see a coordinate vector (a, b, c) .

An analogy

Let's start with a simple example of a basis. If you look at the HTML source code behind any web page, you're sure to find at least one mention of the colour stylesheet directive such as `color:#336699;`. The numbers should be interpreted as a triple of values (33, 66, 99), each value describing the amount of red, green, and blue needed to

The right-hand rule

Consider the plane formed by the vectors \vec{a} and \vec{b} . There are actually *two* vectors perpendicular to this plane: one above the plane and one below the plane. We use the *right-hand rule* to figure out which of these vectors corresponds to the cross product $\vec{a} \times \vec{b}$.

Make a fist with your right hand and then extend your thumb, first finger, and middle finger. When your index finger points in the same direction as the vector \vec{a} and your middle finger points in the direction of \vec{b} , your thumb will point in the direction of $\vec{a} \times \vec{b}$. The relationship encoded in the right-hand rule matches the relationship between the standard basis vectors: $\hat{i} \times \hat{j} = \hat{k}$.

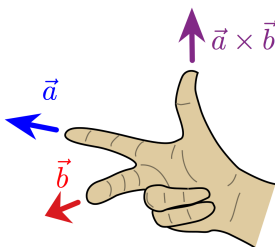


Figure 7.12: Using the right-hand rule to determine the direction of the cross product $\vec{a} \times \vec{b}$ based on directions of \vec{a} and \vec{b} .

Links

[Nice illustrations of the cross product]

<http://1ucasvb.tumblr.com/post/76812811092/>

<https://www.youtube.com/watch?v=eu6i7WJeiw>

Exercises

E7.4 Given the vectors $\vec{u} = (1, 1, 0)$ and $\vec{v} = (0, 0, 3)$, compute the following vector expressions:

a) $\vec{u} + \vec{v}$

b) $\vec{u} - \vec{v}$

c) $3\vec{u} + \vec{v}$

d) $\|\vec{u}\|$

E7.5 Given $\vec{v} = (1, 2, 3)$ and $\vec{w} = (0, 1, 1)$, compute the following vector products: a) $\vec{v} \cdot \vec{w}$; b) $\vec{v} \times \vec{w}$; c) $\vec{w} \times \vec{v}$; d) $\vec{w} \times \vec{w}$.

7.5 Complex numbers

By now, you've heard about complex numbers \mathbb{C} . The word "complex" is an intimidating word. Surely it must be a complex task to

Chapter 8

Extra topics

Learning all of mathematics would require several lifetimes. There is so much to learn: applied math, abstract theoretical math, numerical methods, and many other subfields and specialties. It's up to you to choose how much you want to learn. As an adult learner there will be no exams to force you to study, so if you're learning something it's because you *want* to know!

Out of the thousands of possible math topics you could learn next, I've selected a shortlist of three important topics to get you started. In Section 8.1 we'll explore how to solve equations with multiple unknowns. Specifically, we'll focus on systems of k linear equations that contain k unknowns. For example, the *system of equations*

$$\begin{aligned}1s + 2t &= 5 \\ 3s + 9t &= 21\end{aligned}$$

consists of two equations that contain two unknowns: s and t . There is a systematic procedure you can follow to combine the equations and reduce the problem to a single equation with a single unknown

$$3t = 6,$$

which you know how to solve. In the end, we find the unknowns s and t that satisfy both equations are $s = 1$ and $t = 2$.

In Section 8.2, we'll learn about the compound interest calculations used by banks to compute the interest owed on loans. It's important to understand the math so you can calculate the cost of borrowing for different interest rates and compounding methods. This topic is particularly relevant for students who are considering taking out student loans. Think of this math knowledge as financial self-defence.

Finally, in Section 8.3, we'll introduce sets and set notation. Mathematicians often use symbols like \in (element of), \subset (subset of), \forall (for all), and \exists (there exists), to make very concise math statements and definitions. We managed to get through the entire book without the need for these "alien symbols," but knowing set notation is important for your future math studies. Most advanced math textbooks assume readers are familiar with these symbols, so it's a good idea to know what the symbols mean.

8.1 Solving systems of linear equations

Solving equations with one unknown—like $2x + 4 = 7x$, for instance—requires manipulating both sides of the equation until the unknown variable is *isolated* on one side. For this instance, we can subtract $2x$ from both sides of the equation to obtain $4 = 5x$, which simplifies to $x = \frac{4}{5}$.

What about the case when you are given *two* equations and must solve for *two* unknowns? For example,

$$\begin{aligned}x + 2y &= 5, \\ 3x + 9y &= 21.\end{aligned}$$

Can you find values of x and y that satisfy both equations?

Concepts

- x, y : the two unknowns in the equations
- *eq1, eq2*: a system of two equations that must be solved *simultaneously*. These equations will look like

$$\begin{aligned}a_1x + b_1y &= c_1, \\ a_2x + b_2y &= c_2,\end{aligned}$$

where as , bs , and cs are given constants.

Principles

If you have n equations and n unknowns, you can solve the equations *simultaneously* and find the values of the unknowns. There are several different approaches for solving equations simultaneously. We'll show three of these approaches for the case $n = 2$.

Geometric solution

Solving a system of two linear equations in two unknowns can be understood geometrically as finding the point of intersection between two lines in the Cartesian plane. In this section we'll explore this correspondence between algebra and geometry to develop yet another way of solving systems of linear equations.

The algebraic equation $ax + by = c$ containing the unknowns x and y can be interpreted as a *constraint* equation on the set of possible values for the variables x and y . We can visualize this constraint geometrically by considering the coordinate pairs (x, y) that lie in the Cartesian plane. Recall that every point in the Cartesian plane can be represented as a coordinate pair (x, y) , where x and y are the coordinates of the point.

Figure 8.1 shows the geometrical representation of three equations. The line ℓ_a corresponds to the set of points (x, y) that satisfy the equation $x = 1$, the line ℓ_b is the set of points (x, y) that satisfy the equation $y = 2$, and the line ℓ_c corresponds to the set of points that satisfy $x + 2y = 2$.

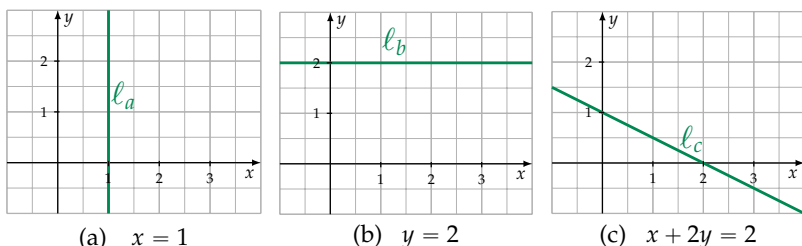


Figure 8.1: Graphical representations of three linear equations.

You can convince yourself that the geometric lines shown in Figure 8.1 are equivalent to the algebraic equations by considering individual points (x, y) in the plane. For example, the points $(1, 0)$, $(1, 1)$, and $(1, 2)$ are all part of the line ℓ_a since they satisfy the equation $x = 1$. For the line ℓ_c , you can verify that the line's x -intercept $(2, 0)$ and its y -intercept $(0, 1)$ both satisfy the equation $x + 2y = 2$.

The Cartesian plane as a whole corresponds to the set \mathbb{R}^2 , which describes all possible pairs of coordinates. To understand the equivalence between the algebraic equation $ax + by = c$ and the line ℓ in the Cartesian plane, we can use the following precise math notation:

$$\ell : \{(x, y) \in \mathbb{R}^2 \mid ax + by = c\}.$$

In words, this means that the line ℓ is defined as the subset of the pairs of real numbers (x, y) that satisfy the equation $ax + by = c$.

Figure 8.2 shows the graphical representation of the line ℓ .

You don't have to take my word for it, though! Think about it and convince yourself that all points on the line ℓ shown in Figure 8.2 satisfy the equation $ax + by = c$. For example, you can check that the x -intercept $(\frac{c}{a}, 0)$ and the y -intercept $(0, \frac{c}{b})$ satisfy the equation $ax + by = c$.

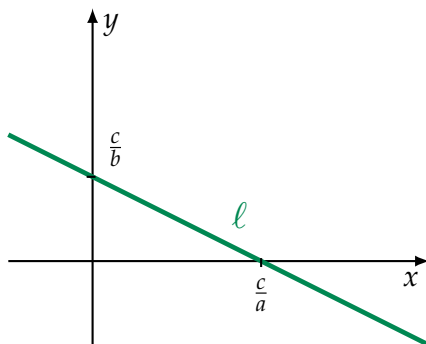


Figure 8.2: Graphical representation of the equation $ax + by = c$.

Solving the system of two equations

$$a_1x + b_1y = c_1,$$

$$a_2x + b_2y = c_2,$$

corresponds to finding the intersection of the lines ℓ_1 and ℓ_2 that represent each equation. The pair (x, y) that satisfies both algebraic equations simultaneously is equivalent to the point (x, y) that is the intersection of lines ℓ_1 and ℓ_2 , as illustrated in Figure 8.3.

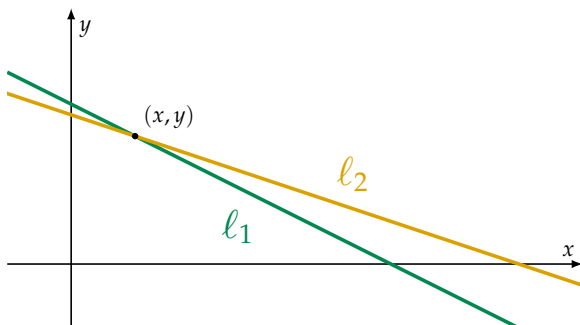


Figure 8.3: The point (x, y) that lies at the intersection of lines ℓ_1 and ℓ_2 .

Example Let's see how we can use the geometric interpretation to solve the system of equations

$$\begin{aligned}x + 2y &= 5, \\ 3x + 9y &= 21.\end{aligned}$$

We've already seen three different *algebraic* techniques for finding the solution to this system of equations; now let's see a *geometric* approach for finding the solution. I'm not kidding you, we're going to solve the exact same system of equations a fourth time!

The first step is to draw the lines that correspond to each of the equations using pen and paper or a graphing calculator. The second step is to find the coordinates of the point where the two lines intersect as shown in Figure 8.4. The point $(1, 2)$ that lies on both lines ℓ_1 and ℓ_2 corresponds to the x and y values that satisfy both equations simultaneously.

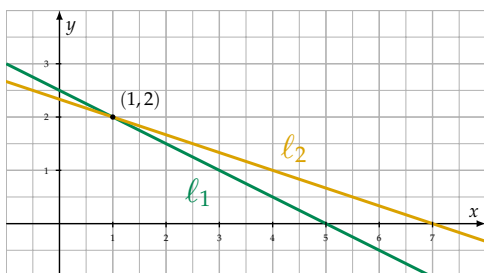


Figure 8.4: The line ℓ_1 with equations $x + 2y = 5$ intersects the line ℓ_2 with equation $3x + 9y = 21$ at the point $(1, 2)$.

Visit the webpage at www.desmos.com/calculator/exikik615f to play with an interactive version of the graphs shown in Figure 8.4. Try changing the equations and see how the graphs change.

Exercises

E8.1 Plot the lines ℓ_a , ℓ_b , and ℓ_c shown in Figure 8.1 (page 165) using the Desmos graphing calculator. Use the graphical representation of these lines to find: **a)** the intersection of lines ℓ_c and ℓ_a , **b)** the intersection of ℓ_a and ℓ_b , and **c)** the intersection of lines ℓ_b and ℓ_c .

E8.2 Solve the system of equations simultaneously for x and y :

$$\begin{aligned}2x + 4y &= 16, \\ 5x - y &= 7.\end{aligned}$$

8.3 Set notation

A *set* is the mathematically precise notion for describing a group of objects. You don't need to know about sets to perform simple math; but more advanced topics require an understanding of what sets are and how to denote set membership, set operations, and set containment relations. This section introduces all the relevant concepts.

Definitions

- *set*: a collection of mathematical objects
- S, T : the usual variable names for sets
- $s \in S$: this statement is read “ s is an element of S ” or “ s is in S ”
- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$: some important number sets: the naturals, the integers, the rationals, and the real numbers, respectively.
- \emptyset : the *empty set* is a set that contains no elements
- $\{ \dots \}$: the curly brackets are used to define sets, and the expression inside the curly brackets describes the set contents.

Set operations:

- $S \cup T$: the *union* of two sets. The union of S and T corresponds to the elements in either S or T .
- $S \cap T$: the *intersection* of the two sets. The intersection of S and T corresponds to the elements that are in both S and T .
- $S \setminus T$: *set difference* or *set minus*. The set difference $S \setminus T$ corresponds to the elements of S that are not in T .

Set relations:

- \subset : is a strict subset of
- \subseteq : is a subset of or equal to

Here is a list of special mathematical shorthand symbols and their corresponding meanings:

- \in : element of
- \notin : not an element of
- \forall : for all
- \exists : there exists
- \nexists : there doesn't exist
- $|$: such that

is not an element of B . Using set notation, the previous sentence is expressed as

$$B \subset A \quad \Leftrightarrow \quad \forall b \in B, b \in A \text{ and } \exists a \in A \text{ such that } a \notin B.$$

For example, the expression $E \subset \mathbb{Z}$ shows that the even numbers are a strict subset of the integers. Every even number is an integer, but there exist integers that are not even (the odd numbers). Some mathematicians prefer the more descriptive symbol \subsetneq to describe strict containment relations.

A subset-or-equal relation is denoted $B \subseteq A$. In writing $B \subseteq A$, a mathematician claims, “Every element of B is also an element of A ,” but makes no claim about the existence of elements that are contained in A but not in B . The statement $B \subset A$ implies $B \subseteq A$; however, $B \subseteq A$ does not imply $B \subset A$. This is analogous to how $b < a$ implies $b \leq a$, but $b \leq a$ doesn’t imply $b < a$, since a and b could be equal.

Set operations

Venn diagrams also help us visualize the subsets obtained from set operations. Figure 8.8 illustrates the set union $A \cup B$, the set intersection $A \cap B$, and the set difference $A \setminus B$, for two sets A and B .

The union $A \cup B$ describes all elements that are in either set A or set B , or both. If $e \in A \cup B$, then $e \in A$ or $e \in B$.

Recall the set of even numbers $E \subset \mathbb{Z}$ and the set of odd numbers $O \subset \mathbb{Z}$ defined above. Since every integer is either an even number or an odd number, we know $\mathbb{Z} \subseteq E \cup O$. The union of two subsets is always contained within the parent set, so we also know $E \cup O \subseteq \mathbb{Z}$. Combining these facts, we can establish the equality $E \cup O = \mathbb{Z}$, which states the fact, “The combination of all even and odd numbers is the same as all integers.”

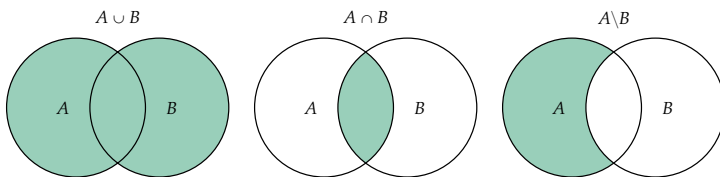


Figure 8.8: Venn diagrams showing different subsets obtained using the set operations: set union $A \cup B$, set intersection $A \cap B$, and set difference $A \setminus B$.

The set intersection $A \cap B$ and set difference $A \setminus B$ are also illustrated in Figure 8.8. The intersection of two sets contains the elements that are part of both sets. The set difference $A \setminus B$ contains all the elements that are in A but not in B .

- The solution set for the equation $\sin(x) = 2$ is \emptyset (the empty set), since there is no number x that satisfies the equation.

The SymPy function `solve` returns the solutions of equations as a list. To solve the equation $f(x) = c$ using SymPy, we first rewrite it as expression that equals zero $f(x) - c = 0$, then call the function `solve`:

```
>>> solve(x-3 -6, x)          # usage: solve(expr, var)
[9]

>>> solve(x**2 -4, x)
[-2, 2]

>>> solve(sin(x), x)
[0, pi]                        # found only solutions in [0,2*pi)

>>> solve(sin(x) -2, x)
[]                              # empty list = empty set
```

In the next section we'll learn how the notion of a solution set is used for describing the solutions to systems of equations.

Solution sets to systems of equations

Let's revisit what we learned in Section 8.1 about the solutions to systems of linear equations, and define their solution sets more precisely. The solution set for the system of equations

$$\begin{aligned}a_1x + b_1y &= c_1, \\a_2x + b_2y &= c_2,\end{aligned}$$

corresponds to the intersection of two sets:

$$\underbrace{\{(x, y) \in \mathbb{R}^2 \mid a_1x + b_1y = c_1\}}_{\ell_1} \cap \underbrace{\{(x, y) \in \mathbb{R}^2 \mid a_2x + b_2y = c_2\}}_{\ell_2}.$$

Recall that the lines ℓ_1 and ℓ_2 are the geometric interpretation of these sets. Each line corresponds to a set of coordinate pairs (x, y) that satisfy the equation of the line. The solution to the system of equations is the set of points at the intersection of the two lines $\ell_1 \cap \ell_2$. Note the word *intersection* is used in two different mathematical contexts: the solution is the *set intersection* of two sets, and also the *geometric intersection* of two lines.

Let's take advantage of this correspondence between set intersections and geometric line intersections to understand the solutions to systems of equations in a little more detail. In the next three sections, we'll look at three possible cases that can occur when trying

to solve a system of two linear equations in two unknowns. So far we've only discussed Case A, which occurs when the two lines intersect at a point, as in the example shown in Figure 8.9. To fully understand the possible solutions to a system of equations, we need to think about all other cases; like Case B when $\ell_1 \cap \ell_2 = \emptyset$ as in Figure 8.10, and Case C when $\ell_1 \cap \ell_2 = \ell_1 = \ell_2$ as in Figure 8.11.

Case A: One solution. When the lines ℓ_1 and ℓ_2 are non-parallel, they will intersect at a point as shown in Figure 8.9. In this case, the solution set to the system of equations contains a single point:

$$\{(x, y) \in \mathbb{R}^2 \mid x + 2y = 2\} \cap \{(x, y) \in \mathbb{R}^2 \mid x = 1\} = \{(1, \tfrac{1}{2})\}.$$

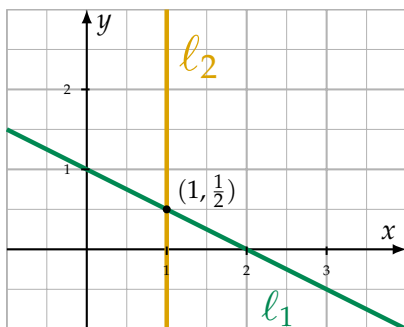


Figure 8.9: Case A: The intersection of the lines with equations $x + 2y = 2$ and $x = 1$ is the point $(1, \frac{1}{2}) \in \mathbb{R}^2$.

Case B: No solution. If the lines ℓ_1 and ℓ_2 are parallel then they will never intersect. The intersection of these lines is the empty set:

$$\{(x, y) \in \mathbb{R}^2 \mid x + 2y = 2\} \cap \{(x, y) \in \mathbb{R}^2 \mid x + 2y = 4\} = \emptyset.$$

Think about it—there is no point (x, y) that lies on both ℓ_1 and ℓ_2 . Using algebra terminology, we say this system of equations has no solution, since there are no numbers x and y that satisfy both equations.

Case C: Infinitely many solutions. If the lines ℓ_1 and ℓ_2 are parallel and overlapping then they intersect everywhere. This case occurs when one of the equations in a system of equations is a multiple of the other equation, as in the case of equations $x + 2y = 2$ and $3x + 6y = 6$. The lines ℓ_1 and ℓ_2 that correspond to these equations are shown in Figure 8.11. Any point (x, y) that satisfies $x + 2y = 2$

Chapter 9

Practice problems

We've now reached the problems chapter in this book. The purpose of these problems is to give you a way to comprehensively practice your math fundamentals. Knowing how to solve math problems is a very useful skill to develop. At times, honing your math chops might seem like tough mental work, but at the end of each problem, you'll gain a stronger foothold on all the topics you've been learning about. You'll also experience a small *achievement buzz* after each problem you vanquish.

Sit down and take a crack at these practice problems today, or another time when you're properly caffeinated. If you make time for some math practice, you'll develop long-lasting comprehension and true math fluency.

Without solving any problems, you're likely to forget most of what you've learned in the next few months. You might still remember the big ideas, but the details will be fuzzy and faded. By solving some of the practice problems, you'll remember a lot more stuff. Don't break the pace now: with math, it's very much *use it or lose it!*

Make sure you put your phone away while you're working on the problems. You don't need fancy technology to do math; grab a pen and some paper from the printer and you'll be fine. The great mathematicians like Descartes, Hilbert, Leibniz, and Noether did most of their work with pen and paper and they did well. Spend some time with math the way they did.

P9.1 Solve for x in the equation $x^2 - 9 = 7$.

P9.2 Solve for x in the equation $\cos^{-1}\left(\frac{x}{A}\right) - \phi = \omega t$.

If you start from an idea $p_0 = 100i$ and the cumulative number of hours invested after t weeks of working on the project is $h(t) = 0.2t^2$, how long will it take for the project to become 100% real? Assume $\alpha = 2.904 \times 10^{-3}$.
Hint: A project is 100% real if $\text{Re}\{p\} = p$.

P9.70 A farmer with a passion for robotics has built a prototype of a robotic tractor. The tractor is programmed to move with a speed of 0.524 km/h and follow the direction of the hour-hand on a conventional watch. Assume the tractor starts at 12:00 p.m. (noon) and is left to roam about in a field until 6:00 p.m. What is the shape of the trajectory that the tractor will follow? What is the total distance travelled by the tractor after six hours?

P9.71 Solve for x and y simultaneously in the following system of equations: $-x - 2y = -2$ and $3x + 3y = 0$.

P9.72 Solve the following system of equations for the three unknowns:

$$\begin{aligned} 1x + 2y + 3z &= 14, \\ 2x + 5y + 6z &= 30, \\ -1x + 2y + 3z &= 12. \end{aligned}$$

P9.73 A hotel offers a 15% discount on rooms. Determine the original price of a room if the discounted room price is \$95.20.

P9.74 A set of kitchen tools normally retails for \$450, but today it is priced at the special offer of \$360. Calculate the percentage of the discount.

P9.75 You take out a \$5000 loan at a nominal annual percentage rate (nAPR) of 12% with monthly compounding. How much money will you owe after 10 years?

P9.76 Plot the graphs of $f(x) = 100e^{-x/2}$ and $g(x) = 100(1 - e^{-x/2})$ by evaluating the functions at different values of x from 0 to 11.

P9.77 Starting from an initial quantity Q_0 of Exponentium at $t = 0$ s, the quantity Q of Exponentium as a function of time varies according to the expression $Q(t) = Q_0 e^{-\lambda t}$, where $\lambda = 5.0$ and t is measured in seconds. Find the *half-life* of Exponentium, that is, the time it takes for the quantity of Exponentium to reduce to half the initial quantity Q_0 .

P9.78 A hot body cools so that every 24 min its temperature decreases by a factor of two. Deduce the time-constant and determine the time it will take the body to reach 1% of its original temperature.

Hint: The temperature function is $T(t) = T_0 e^{-t/\tau}$ and τ is the *time constant*.

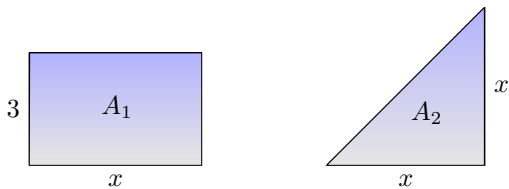
P9.79 A capacitor of capacitance $C = 4.0 \times 10^{-6}$ farads, charged to an initial potential of $V_0 = 20$ volts, is discharging through a resistance of $R = 10\,000\,\Omega$ (read Ohms). Find the potential V after 0.01 s and after 0.1 s, knowing the decreasing potential follows the rule $V(t) = V_0 e^{-\frac{t}{RC}}$.

P9.80 Let B be the set of people who are bankers and C be the set of crooks. Rewrite the math statement $\exists b \in B \mid b \notin C$ in plain English.

P9.81 Let M denote the set of people who run Monsanto, and H denote the people who ought to burn in hell for all eternity. Write the math statement $\forall p \in M, p \in H$ in plain English.

P9.82 When starting a business, one sometimes needs to find investors. Define M to be the set of investors with money, and C to be the set of investors with connections. Describe the following sets in words: **a)** $M \setminus C$, **b)** $C \setminus M$, and the most desirable set **c)** $M \cap C$.

P9.83 Write the formulas for the functions $A_1(x)$ and $A_2(x)$ that describe the areas of the following geometric shapes.



End matter

Conclusion

We covered many fundamental concepts during this hike in the math mountains. We started with basic math ideas like variables, expressions, and equations, and then moved into algebra—hopefully you’ll agree with me that algebra is not as scary as some people think, right? We covered many other scary topics like exponents and logarithms, and you made it through unscathed. We even tackled more advanced topics like functions, vectors, complex numbers, proofs, and set notation. Congratulations on powering through this intensive math bootcamp!

The only way to truly understand math is to apply math concepts to solving problems. You don’t get points for memorizing rules, formulas, and equations; only when you learn how to *use* the equations will you truly “own” the math. Math is not a spectator sport! If you’re reading this book without solving the practice problems, you’re getting a bad deal. If you haven’t attempted to solve the problems in Chapter 9, you need to turn back and give them a shot.

My intention with this book was to give you a solid, basic understanding of many mathematical concepts, explained quickly and simply so you wouldn’t get bored. However, it’s possible that the book’s fast pace led to some gaps in the coverage of the material. Let me know if you find any parts of the book that lack clarity or are missing explanations. Send me an email if you find a confusing explanation, a missing definition, or an equation you think is wrong. You can reach me at ivan@minireference.com. The book’s relative errorlessness is thanks to the many typo fixes, bug reports, and helpful suggestions I receive from readers like you.

To learn more about other books in the **No Bullshit Guide** series and find out how Minireference Co. is revolutionizing the textbook industry, visit our blog at minireference.com/blog. You can also find us on the twitter [@minireference](https://twitter.com/minireference) and on the facebook fb.me/noBSguide.

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Last but not least, I want to thank all my students for their endless questions, demand for explanations, and curiosity. If I've developed any ability to explain things, I owe it to them.

Further reading

You've reached the end of this book, but you're only at the beginning of the journey of mathematical discovery. There are many things left for you to learn about. Below are some recommendations of subjects you might find interesting.

Newtonian mechanics

Newton's laws of physics provide an excellent example of the scientific approach to understanding and modelling the world around us. Using a few basic laws, it's possible to build accurate models for the motion of objects in terms of concepts like position, velocity, acceleration, momentum, and energy. It's no surprise mechanics is a foundational subject of science.

Since you know basic math, it will be easy for you to understand the laws of physics. Indeed, most physics laws are expressed as mathematical equations. If you know how to manipulate equations and solve for the unknowns in them, then you already know half of physics. If you previously avoided this subject out of fear, now would be a great time to revisit it. Learning physics is an excellent source of *knowledge buzz*.

By learning how to solve complicated physics problems, you'll also develop your analytical skills. Later, you can apply these skills to other areas of life. Even if you don't go on to study science, the expertise you develop in solving physics problems will help you tackle complicated problems in general.

[TUTORIAL] Ivan Savov. *Mechanics explained in seven pages*, 2017, see minireference.com/static/tutorials/mech_in_7_pages.pdf.

[BOOK] Ivan Savov. *No Bullshit Guide to Math & Physics*, Minireference Publishing, Fifth edition, 2014, 978-0-9920010-0-1.

Calculus

Calculus is the language of quantitative science. Many of the fundamental laws of physics, chemistry, biology, and economics are expressed in the language of calculus. A solid grasp of this subject is essential for anyone interested in pursuing university-level studies in science, engineering, or business.

The new knowledge you'll develop in calculus involves analyzing the properties of functions. You'll learn to describe how functions change over time (derivatives), and how to calculate the total amount of a quantity that accumulates over a time period (integrals). The language of calculus will allow you to speak precisely about the different properties of functions, and better understand their behaviour. You will learn how to calculate the slopes of functions, how to find their maximum and minimum values, how to compute their integrals, and other tasks of practical importance. In addition to its practical applications, learning calculus will also expose you to the concept of infinity. Calculus thinking requires understanding infinitely small numbers, infinitely large numbers, and computations

with an infinite number of steps.

If you liked the math in this book, I encourage you to continue your journey with the **No Bullshit Guide to Math & Physics**, which will teach you the topics of mechanics, differential calculus, and integral calculus in an integrated manner, highlighting the connections between them, and discussing their applications.

[BOOK] Ivan Savov. *No Bullshit Guide to Math & Physics*, Minireference Publishing, Fifth edition, 2014, 978-0-9920010-0-1.

The calculus book by Thompson is also worth checking out.

[BOOK] Silvanus P. Thompson. *Calculus Made Easy*, Macmillian and Co., Second edition, 1914, <http://gutenberg.org/ebooks/33283>.

You can also learn calculus from these excellent video lessons.

[VIDEOS] Grant Sanderson (3Blue1Brown), *Essence of calculus*, YouTube playlist, 2018, online: <https://bit.ly/3Blue1BrownCalc>.

[VIDEO LECTURES] Gilbert Strang. *Highlights of Calculus*, MIT OpenCourseWare, 2010, see <https://bit.ly/StrangCalcLectures>.

Linear algebra

Learning linear algebra will open many doors for you. You need linear algebra to understand statistics, computer graphics, machine learning, quantum mechanics, and many other areas of science and business.

Here's a little preview. Linear algebra is the study of vectors $\vec{v} \in \mathbb{R}^n$ and linear transformations $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$. Linear transformations are *vector functions* that obey the linear property $T(\alpha\vec{v}_1 + \beta\vec{v}_2) = \alpha T(\vec{v}_1) + \beta T(\vec{v}_2)$. Using the standard notation for functions, we write $T(\vec{x}) = \vec{y}$ to show the linear transformation T acting on an input vector $\vec{x} \in \mathbb{R}^n$ to produce the output vector $\vec{y} \in \mathbb{R}^m$. Every linear transformation T can be *represented* as a matrix $A_T \in \mathbb{R}^{m \times n}$, which is an array of numbers with m rows and n columns. Computing $T(\vec{x})$ is equivalent to computing the matrix-vector product $A_T \vec{x}$. Because of the equivalence between linear transformations and matrices, we can also say that linear algebra is the study of vectors and matrices.

Vectors and matrices are used all over the place! If your knowledge of high-school math gave you modelling superpowers, then linear algebra is the vector-upgrade that teaches you how to build models in multiple dimensions.

[VIDEO LECTURES] Gilbert Strang. *Linear Algebra*, MIT OpenCourseWare, 2010, online: <http://bit.ly/StrangLALectures>.

[BOOK] Ivan Savov. *No Bullshit Guide to Linear Algebra*, Minireference Publishing, Second edition, 2017, ISBN 978-0-9920010-2-5.

Probability

Probability distributions are a fundamental tool for modelling non-deterministic behaviour. A discrete random variable X is associated with a probability mass function $p_X(x) \stackrel{\text{def}}{=} \Pr(\{X = x\})$, which assigns a “probability mass” to each of the possible outcomes of the random variable X . For example, if X represents the outcome of the throw of a fair die, then the possible outcomes are $\mathcal{X} = \{1, 2, 3, 4, 5, 6\}$ and the probability mass function has the values $p_X(x) = \frac{1}{6}, \forall x \in \mathcal{X}$.

Probability distributions and random variables allow us to model random processes like the roll of a die. We can’t predict the exact outcome when two dice X_1 and X_2 are rolled, but we can predict the probability of different outcomes. For example, the “pair of sixes” outcome is described by the event $\{X_1 + X_2 = 12\}$. Assuming the dice are fair, this outcome has probability $\Pr(\{X_1 + X_2 = 12\}) = \frac{1}{36}$.

Probability theory is used all over the place, including in statistics, machine learning, quantum mechanics, and risk analysis.

[BOOK] Charles M. Grinstead and J. Laurie Snell. *Introduction to Probability*, AMS, 2006, <https://bit.ly/GrinsteadSnellProb>.

General mathematics

Mathematics is a hugely broad field. There are all kinds of topics to learn about; some of them are fun, some of them are useful, and some of them are totally mind expanding.

The following resources cover math topics of general interest and serve as a great overview of all areas of mathematics. I highly recommend you take a look at these for further math enlightening.

[VIDEO] A map of mathematics that shows all the subfields of mathematics and their objects of study: <https://youtu.be/0mJ-4B-mS-Y>.

[VIDEOS] Video interviews and lessons by some of the best math educators in the world: <https://youtube.com/user/numberphile>.

[BOOK] Alfred North Whitehead. *An Introduction to Mathematics*, Williams & Norgate, 1911, www.gutenberg.org/ebooks/41568.

General physics

If you want to learn more about physics, I highly recommend the Feynman lectures on physics. This three-tome collection covers all of undergraduate physics and explains many more advanced topics.

[BOOK] Richard P. Feynman. *The Feynman Lectures on Physics, The Definitive and Extended Edition*, Addison Wesley, 2005, ISBN 0805390456. Read online at: <http://feynmanlectures.caltech.edu>

Appendix A

Answers and solutions

Chapter 1 solutions

Answers to exercises

E1.1 a) $x = 3$; b) $x = 30$; c) $x = 2$; d) $x = -3$. **E1.2** a) $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$; b) \mathbb{C} ; c) $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$; d) $\mathbb{Q}, \mathbb{R}, \mathbb{C}$; e) \mathbb{R}, \mathbb{C} . **E1.3** a) 21; b) 0; c) $\frac{2}{27}$. **E1.4** a) $\frac{5}{6}$; b) $\frac{13}{12} = 1\frac{1}{12}$; c) $\frac{31}{6} = 5\frac{1}{6}$. **E1.5** a) $x = 2$; b) $x = 25$; c) $x = 100$. **E1.6** a) $f^{-1}(x) = x^2, x = 16$. b) $g^{-1}(x) = -\frac{1}{2}\ln(x), x = 0$.

Solutions to selected exercises

E1.4 a) To compute $\frac{1}{2} + \frac{1}{3}$, we rewrite both fractions using the common denominator 6, then compute the sum: $\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$. b) You can use the answer from part (a), or compute the triple sum directly by setting all three fractions to a common denominator: $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{6}{12} + \frac{4}{12} + \frac{3}{12} = \frac{13}{12}$. c) Here we first rewrite $3\frac{1}{2}$ as $\frac{7}{2}$, then use the common denominator 6 for the computation: $\frac{7}{2} + 2 - \frac{1}{3} = \frac{21}{6} + \frac{12}{6} - \frac{2}{6} = \frac{31}{6}$.

Chapter 2 solutions

Answers to exercises

E2.1 a) $(x-1)(x-7)$; b) $(x+2)^2$; c) $(x+3)(x-3)$. **E2.2** a) $x^2 + 2x - 15 = (x+1)^2 - 16 = 0$, which has solutions $x = 3$ and $x = -5$; b) $x^2 + 4x + 1 = (x+2)^2 - 3 = 0$, with solutions $x = -2 + \sqrt{3}$ and $x = -2 - \sqrt{3}$. **E2.3** $x_1 = \frac{3}{2}$ and $x_2 = -1$. **E2.4** $x = \pm\sqrt{2}$.

Chapter 3 solutions

Answers to exercises

E3.1 a) 8; b) $a^{-1}b^{-2}c^{-3} = \frac{1}{ab^2c^3}$; c) $8a^2$; d) a^6b^{-2} . **E3.2** a) 3; b) 12; c) $\sqrt{3}$; d) $|a|$. **E3.3**

Appendix B

Notation

This appendix contains a summary of the notation used in this book.

Math notation

Expression	Read as	Used to denote
a, b, x, y		variables
$=$	is equal to	expressions that have the same value
$\stackrel{\text{def}}{=}$	is defined as	a new variable definition
$a + b$	a plus b	the combined lengths of a and b
$a - b$	a minus b	the difference in lengths between a and b
$a \times b = ab$	a times b	the area of a rectangle
$a^2 = aa$	a squared	the area of a square of side length a
$a^3 = aaa$	a cubed	the volume of a cube of side length a
a^n	a to the n	a multiplied by itself n times
$\sqrt{a} = a^{\frac{1}{2}}$	square root of a	the side length of a square of area a
$\sqrt[3]{a} = a^{\frac{1}{3}}$	cube root of a	the side length of a cube with volume a
$a/b = \frac{a}{b}$	a divided by b	a parts of a whole split into b parts
$a^{-1} = \frac{1}{a}$	one over a	division by a
$f(x)$	f of x	the function f applied to input x
f^{-1}	f inverse	the inverse function of $f(x)$
$f \circ g$	f compose g	function composition; $f \circ g(x) = f(g(x))$
e^x	e to the x	the exponential function base e
$\ln(x)$	natural log of x	the logarithm base e
a^x	a to the x	the exponential function base a
$\log_a(x)$	log base a of x	the logarithm base a
θ, ϕ	<i>theta, phi</i>	angles
\sin, \cos, \tan	\sin, \cos, \tan	trigonometric ratios
$\%$	percent	proportions of a total; $a\% = \frac{a}{100}$

Set notation

You don't need a lot of fancy notation to understand mathematics. It really helps, though, if you know a little bit of set notation.

Symbol	Read as	Denotes
$\{ \dots \}$	the set ...	define a sets
$ $	such that	describe or restrict the elements of a set
\mathbb{N}	the naturals	the set $\mathbb{N} \stackrel{\text{def}}{=} \{0, 1, 2, \dots\}$. Note $\mathbb{N}^* \stackrel{\text{def}}{=} \mathbb{N} \setminus \{0\}$.
\mathbb{Z}	the integers	the set $\mathbb{Z} \stackrel{\text{def}}{=} \{\dots, -2, -1, 0, 1, 2, 3, \dots\}$
\mathbb{Q}	the rationals	the set of fractions of integers
\mathbb{R}	the reals	the set of real numbers
\mathbb{C}		the set of complex numbers
\subset	subset	one set strictly contained in another
\subseteq	subset or equal	containment or equality
\cup	union	the combined elements from two sets
\cap	intersection	the elements two sets have in common
$S \setminus T$	S set minus T	the elements of S that are not in T
$a \in S$	a in S	a is an element of set S
$a \notin S$	a not in S	a is not an element of set S
$\forall x$	for all x	a statement that holds for all x
$\exists x$	there exists x	an existence statement
$\nexists x$	there doesn't exist x	a non-existence statement

An example of a condensed math statement that uses set notation is " $\nexists m, n \in \mathbb{Z}$ such that $\frac{m}{n} = \sqrt{2}$," which reads "there don't exist integers m and n whose ratio equals $\sqrt{2}$." Since we identify the set of fractions of integers with the rationals, this statement is equivalent to the shorter " $\sqrt{2} \notin \mathbb{Q}$," which reads " $\sqrt{2}$ is irrational."

Complex numbers notation

Expression	Denotes
\mathbb{C}	the set of complex numbers $\mathbb{C} \stackrel{\text{def}}{=} \{a + bi \mid a, b \in \mathbb{R}\}$
i	the unit imaginary number $i \stackrel{\text{def}}{=} \sqrt{-1}$ or $i^2 = -1$
$\text{Re}\{z\} = a$	real part of $z = a + bi$
$\text{Im}\{z\} = b$	imaginary part of $z = a + bi$
$ z \angle \varphi_z$	polar representation of $z = z \cos \varphi_z + i z \sin \varphi_z$
$ z = \sqrt{a^2 + b^2}$	magnitude of $z = a + bi$
$\varphi_z = \tan^{-1}(b/a)$	phase or argument of $z = a + bi$
$\bar{z} = a - bi$	complex conjugate of $z = a + bi$

Vectors notation

Expression	Denotes
\mathbb{R}^n	the set of n -dimensional vectors
\vec{v}	a vector
(v_x, v_y)	vector in component notation
$v_x \hat{i} + v_y \hat{j}$	vector in unit vector notation
$\ \vec{v}\ \angle \theta$	vector in length-and-direction notation
$\ \vec{v}\ $	length of the vector \vec{v}
θ	angle the vector \vec{v} makes with the x -axis
$\hat{v} \stackrel{\text{def}}{=} \frac{\vec{v}}{\ \vec{v}\ }$	unit vector in the same direction as \vec{v}
$\vec{u} \cdot \vec{v}$	dot product of the vectors \vec{u} and \vec{v}
$\vec{u} \times \vec{v}$	cross product of the vectors \vec{u} and \vec{v}

Appendix C

SymPy tutorial

Computers can be very useful for dealing with complicated math expressions or when slogging through tedious calculations. Throughout this book we used SymPy to illustrate several math concepts. We'll now review all the math tools available through the SymPy command line. Don't worry if you're not a computer person; we'll only discuss concepts we covered in the book, and the computer commands we'll learn are very similar to the math operations you're already familiar with. This section also serves as a final review of the material covered in the book.

Introduction

You can use a computer algebra system (CAS) to manipulate complicated math expressions and solve any equation.

All computer algebra systems offer essentially the same functionality, so it doesn't matter which system you use: there are free systems like SymPy, Magma, or Octave, and commercial systems like Maple, MATLAB, and Mathematica. This tutorial is an introduction to SymPy, which is a *symbolic* computer algebra system written in the programming language Python. In a symbolic CAS, numbers and operations are represented symbolically, so the answers obtained are exact. For example, the number $\sqrt{2}$ is represented in SymPy as the object `Pow(2, 1/2)`, whereas in *numerical* computer algebra systems like Octave, the number $\sqrt{2}$ is represented as the approximation 1.41421356237310 (a float). For most purposes the approximation is okay, but sometimes approximations can lead to problems: `float(sqrt(2))*float(sqrt(2)) = 2.00000000000000044` \neq 2. Because SymPy uses exact representations, you'll never run into such problems: `Pow(2, 1/2)*Pow(2, 1/2) = 2`.

This tutorial presents many explanations as code snippets. Be sure to try the code examples on your own by typing the commands into SymPy. It's always important to verify for yourself!

Using SymPy

The easiest way to use SymPy, provided you're connected to the internet, is to visit <http://live.sympy.org>. You'll be presented with an interactive prompt into which you can enter your commands—right in your browser.

If you want to use SymPy on your own computer, you must first install Python and the Python package `sympy`. You can then open a command prompt and start a Python session using:

```
you@host> python
Python X.Y.Z
[GCC a.b.c (Build Info)] on platform
Type "help", "copyright", or "license" for more information.
>>>
```

The `>>>` prompt indicates you're in the Python shell which accepts Python commands. Type the following in the Python shell:

```
>>> from sympy import *
>>>
```

The command `from sympy import *` imports all the SymPy functions into the current namespace. All SymPy functions are now available to you. To exit the python shell press CTRL+D.

For an even better experience, you can try `jupyter notebook`, which is a web interface for accessing the Python shell. Search the web for “jupyter notebook” and follow the installation instructions specific to your operating system. It's totally worth it!

Each section in this appendix begins with a `python import` statement for the functions used in that section. If you use the statement `from sympy import *` in the beginning of your code, you don't need to run these individual import statements, but I've included them so you'll know which SymPy vocabulary is covered in each section.

Fundamentals of mathematics

Let's begin by learning about the basic SymPy objects and the operations we can carry out on them. We'll learn the SymPy equivalents of many math verbs like: “to solve” (an equation), “to expand” (an expression), “to factor” (a polynomial).

Numbers

```
>>> from sympy import sympify, S, evalf, N
```

In Python, there are two types of number objects: ints and floats.

```
>>> 3
3                                # an int
>>> 3.0
3.0                              # a float
```

Integer objects in Python are a faithful representation of the set of integers $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$. Floating point numbers are approximate representations of the reals \mathbb{R} . A floating point number has 16 decimals of precision.

Special care is required when specifying rational numbers if you want to get exact answers. If you try to divide two numbers, Python will compute a floating point approximation:

```
>>> 1/7
0.14285714285714285              # a float
```

The floating point number 0.14285714285714285 is an approximation of the exact number $\frac{1}{7} \in \mathbb{Q}$. The float approximation has 16 decimals while the decimal expansion of $\frac{1}{7}$ is infinitely long. To obtain an *exact* representation of $\frac{1}{7}$ you need to create a SymPy expression. You can `sympify` any expression using the shortcut function `S()`:

```
>>> S('1/7')                    # = sympify('1/7')
1/7                              # = Rational(1,7)
```

Note the input to `S()` is specified as a text string delimited by quotes. We could have achieved the same result using `S('1')/7` since a SymPy object divided by an int is a SymPy object.

Except for the tricky Python division operator, other math operators like addition `+`, subtraction `-`, and multiplication `*` work as you would expect. The syntax `**` is used to denote exponentiation:

```
>>> 2**10                        # same as S('2^10')
1024
```

When solving math problems, it's best to work with SymPy objects, and wait to compute the numeric answer in the end. To obtain a numeric approximation of a SymPy object as a float, call its `.evalf()` method:

```
>>> pi
pi
>>> pi.evalf()                  # = pi.n() = N(pi)
3.14159265358979
```

The method `.n()` is equivalent to `.evalf()`. The global SymPy function `N()` can also be used to compute numerical values. You can easily change the number of digits of precision of the approximation. Enter `pi.n(400)` to obtain an approximation of π to 400 decimals.

Symbols

```
>>> from sympy import Symbol, symbols
```

Python is a civilized language so there's no need to define variables before assigning values to them. When you write `a = 3`, you define a new name `a` and set it to the value 3. You can now use the name `a` in subsequent calculations.

Most interesting SymPy calculations require us to define symbols, which are the SymPy objects for representing variables and unknowns. For your convenience, when `live.sympy.org` starts, it runs the following commands automatically:

```
>>> from sympy import *
>>> x, y, z, t = symbols('x y z t')
>>> k, m, n = symbols('k m n', integer=True)
>>> f, g, h = symbols('f g h', cls=Function)
```

The first statement imports all the SymPy functions. The other three statements define some generic symbols `x`, `y`, `z`, and `t`, and several other symbols with special properties.

Note the difference between the following two statements:

```
>>> x + 2
x + 2                # an Add expression
>>> p + 2
NameError: name 'p' is not defined
```

The name `x` is defined as a symbol, so SymPy knows that `x + 2` is an expression; but the variable `p` is not defined, so SymPy doesn't know what to make of `p + 2`. To use `p` in expressions, you must first define it as a symbol:

```
>>> p = Symbol('p')    # the same as p = symbols('p')
>>> p + 2
p + 2                  # = Add(Symbol('p'), Integer(2))
```

You can define a sequence of variables using the following notation:

```
>>> a0, a1, a2, a3 = symbols('a0:4')
```

You can use any name you want for a variable, but it's best if you avoid the letters `Q`, `C`, `O`, `S`, `I`, `N` and `E` because they have special uses in SymPy: `I` is the unit imaginary number $i \stackrel{\text{def}}{=} \sqrt{-1}$, `E` is the base of the natural logarithm, `S()` is the `sympify` function, `N()` is used to obtain numeric approximations, and `O` is used for big- O notation.

Expressions

```
>>> from sympy import simplify, factor, expand, collect
```

You define SymPy expressions by combining symbols with basic math operations and other functions:


```
>>> expr = 2*x + 3*x - sin(x) - 3*x + 42
>>> simplify(expr)           # simplify the expression
2*x - sin(x) + 42
```

The function `simplify` can be used on any expression to simplify it. The examples below illustrate other useful SymPy functions that correspond to common mathematical operations on expressions:

```
>>> factor( x**2-2*x-8 )      # factor a polynomial
(x - 4)*(x + 2)
>>> expand( (x-4)*(x+2) )     # expand an expression
x**2 - 2*x - 8
>>> collect(x**2+x*b+a*x+a*b, x) # collect like terms in x
x**2 + (a+b)*x + a*b
```

To substitute a given value into an expression, call the `.subs()` method, passing in a python dictionary object `{ key:val, ... }` with the symbol–value substitutions you want to make:

```
>>> expr = sin(x) + cos(y)    # define an expression
>>> expr
sin(x) + cos(y)
>>> expr.subs({x:1, y:2})     # subs. x=1,y=1 in expr
sin(1) + cos(2)
>>> expr.subs({x:1, y:2}).n() # compute numeric value
0.425324148260754
```

Note how we used `.n()` to obtain the expression’s numeric value.

Solving equations

```
>>> from sympy import solve
```

The function `solve` is the main workhorse in SymPy. This incredibly powerful function knows how to solve all kinds of equations. In fact `solve` can solve pretty much *any* equation! When high school students learn about this function, they get really angry—why did they spend five years of their life learning to solve various equations by hand, when all along there was this `solve` thing that could do all the math for them? Don’t worry, learning math is *never* a waste of time.

The function `solve` takes two arguments. Use `solve(expr, var)` to solve the equation $\text{expr}=0$ for the variable `var`. You can rewrite any equation in the form $\text{expr}=0$ by moving all the terms to one side of the equation; the solutions to $A(x) = B(x)$ are the same as the solutions to $A(x) - B(x) = 0$.

For example, to solve the quadratic equation $x^2 + 2x - 8 = 0$, use

```
>>> solve( x**2 + 2*x - 8, x)
[2, -4]
```

In this case the equation has two solutions so `solve` returns a list. Check that $x = 2$ and $x = -4$ satisfy the equation $x^2 + 2x - 8 = 0$.

The best part about `solve` and SymPy is that you can obtain symbolic answers when solving equations. Instead of solving one specific quadratic equation, we can solve all possible equations of the form $ax^2 + bx + c = 0$ using the following steps:

```
>>> a, b, c = symbols('a b c')
>>> solve( a*x**2 + b*x + c, x)
[(-b+sqrt(b**2 - 4*a*c))/(2*a), (-b-sqrt(b**2-4*a*c))/(2*a)]
```

In this case `solve` calculated the solution in terms of the symbols a , b , and c . You should be able to recognize the expressions in the solution—it's the quadratic formula $x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

To solve a *system of equations*, you can feed `solve` with the list of equations as the first argument, and specify the list of unknowns you want to solve for as the second argument. For example, to solve for x and y in the system of equations $x + y = 3$ and $3x - 2y = 0$, use

```
>>> solve([x + y - 3, 3*x - 2*y], [x, y])
{x: 6/5, y: 9/5}
```

The function `solve` is like a Swiss Army knife you can use to solve all kind of problems. Suppose you want to *complete the square* in the expression $x^2 - 4x + 7$, that is, you want to find constants h and k such that $x^2 - 4x + 7 = (x - h)^2 + k$. There is no special “complete the square” function in SymPy, but you can call `solve` on the equation $(x - h)^2 + k - (x^2 - 4x + 7) = 0$ to find the unknowns h and k :

```
>>> h, k = symbols('h k')
>>> solve( (x-h)**2 + k - (x**2-4*x+7), [h,k] )
[(2, 3)]                                     # so h = 2 and k = 3
>>> expand( (x-2)**2+3 )                     # verify...
x**2 - 4*x + 7
```

Learn the basic SymPy commands and you'll never need to suffer another tedious arithmetic calculation painstakingly performed by hand again!

Rational functions

```
>>> from sympy import together, apart
```

By default, SymPy will not combine or split rational expressions. You need to use `together` to symbolically calculate the addition of fractions:

```
>>> a, b, c, d = symbols('a b c d')
>>> a/b + c/d
a/b + c/d
>>> together(a/b + c/d)
(a*d + b*c)/(b*d)
```

Alternately, if you have a rational expression and want to divide the numerator by the denominator, use the `apart` function:

```
>>> apart( (x**2+x+4)/(x+2) )
x - 1 + 6/(x + 2)
```

Exponentials and logarithms

Euler's number $e = 2.71828\dots$ is defined one of several ways,

$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = \lim_{\epsilon \rightarrow 0} (1 + \epsilon)^{1/\epsilon} = \sum_{n=0}^{\infty} \frac{1}{n!},$$

and is denoted `E` in SymPy. Using `exp(x)` is equivalent to `E**x`.

The functions `log` and `ln` both compute the logarithm base e :

```
>>> log(E**3)    # same as ln(E**3)
3
```

By default, SymPy assumes the inputs to functions like `exp` and `log` are complex numbers, so it will not expand certain logarithmic expressions. However, indicating to SymPy that the inputs are positive real numbers will make the expansions work:

```
>>> x, y = symbols('x y')
>>> expand( log(x*y) )
log(x*y)
>>> a, b = symbols('a b', positive=True)
>>> expand( log(a*b) )
log(a) + log(b)
```

Polynomials

Let's define a polynomial P with roots at $x = 1$, $x = 2$, and $x = 3$:

```
>>> P = (x-1)*(x-2)*(x-3)
>>> P
(x - 1)*(x - 2)*(x - 3)
```

To see the expanded version of the polynomial, call its `expand` method:

```
>>> expand(P)
x**3 - 6*x**2 + 11*x - 6
```

When the polynomial is expressed in it's expanded form $P(x) = x^3 - 6^2 + 11x - 6$, we can't immediately identify its roots. This is why the factored form $P(x) = (x - 1)(x - 2)(x - 3)$ is preferable. To factor a polynomial, call its `factor` method or `simplify` it:

```
>>> factor(P)
(x - 1)*(x - 2)*(x - 3)
>>> simplify(P)
(x - 1)*(x - 2)*(x - 3)
```

Recall that the roots of the polynomial $P(x)$ are defined as the solutions to the equation $P(x) = 0$. We can use the `solve` function to find the roots of the polynomial:

```
>>> roots = solve(P,x)
>>> roots
[1, 2, 3]
# let's check if P equals (x-1)(x-2)(x-3)
>>> simplify( P - (x-roots[0])*(x-roots[1])*(x-roots[2]) )
0
```

Equality checking

In the last example, we used the `simplify` function on the difference of two expressions to check whether they were equal. This way of checking equality works because $P = Q$ if and only if $P - Q = 0$. To know whether $P = Q$, we can calculate `simplify(P-Q)` and see if the result equals 0. This is the best way to check whether two expressions are equal in SymPy because it attempts all possible simplifications when comparing the expressions. Below is a list of other ways to check whether two quantities are equal, with example cases where equality fails to be detected:

```
>>> P = (x-5)*(x+5)
>>> Q = x**2 - 25
>>> P == Q                                     # fail
False
>>> P - Q == 0                                 # fail
False
>>> simplify(P - Q)                           # works!
0
>>> sin(x)**2 + cos(x)**2 == 1                 # fail
False
>>> simplify( sin(x)**2 + cos(x)**2 - 1 )      # works!
0
```

Trigonometry

```
from sympy import sin, cos, tan, trigsimp, expand_trig
```

The trigonometric functions `sin` and `cos` take inputs in radians:

```
>>> sin(pi/6)
1/2
>>> cos(pi/6)
sqrt(3)/2
```

For angles in degrees, you need a conversion factor of $\frac{\pi}{180}$ [rad/°]:

```
>>> sin(30*pi/180)          # 30 deg = pi/6 rads
1/2
```

The inverse trigonometric functions $\sin^{-1}(x) = \arcsin(x)$ and $\cos^{-1}(x) = \arccos(x)$ are used as follows:

```
>>> asin(1/2)
pi/6
>>> acos(sqrt(3)/2)
pi/6
```

Recall that $\tan(x) = \frac{\sin(x)}{\cos(x)}$. The inverse function of $\tan(x)$ is $\tan^{-1}(x) = \arctan(x) = \text{atan}(x)$

```
>>> tan(pi/6)
1/sqrt(3)          # = ( 1/2 )/( sqrt(3)/2 )
>>> atan( 1/sqrt(3) )
pi/6
```

The function `acos` returns angles in the range $[0, \pi]$, while `asin` and `atan` return angles in the range $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Here are some trigonometric identities that SymPy knows:

```
>>> sin(x) == cos(x - pi/2)
True
>>> simplify( sin(x)*cos(y)+cos(x)*sin(y) )
sin(x + y)
>>> e = 2*sin(x)**2 + 2*cos(x)**2
>>> trigsimp(e)
2
>>> trigsimp(log(e))
log(2*sin(x)**2 + 2*cos(x)**2)
>>> trigsimp(log(e), deep=True)
log(2)
>>> simplify( sin(x)**4 - 2*cos(x)**2*sin(x)**2 + cos(x)**4 )
cos(4*x)/2 + 1/2
```

The function `trigsimp` does essentially the same job as `simplify`.

If instead of simplifying you want to expand a trig expression, you should use `expand_trig`, because the default `expand` won't touch trig functions:

```
>>> expand(sin(2*x))
sin(2*x)
>>> expand_trig(sin(2*x))    # = expand(sin(2*x), trig=True)
2*sin(x)*cos(x)
```

Complex numbers

```
>>> from sympy import I, re, im, Abs, arg, conjugate
```

Consider the quadratic equation $x^2 = -1$. There are no real solutions to this equation, but we can define an imaginary number $i = \sqrt{-1}$ (denoted `I` in SymPy) that satisfies this equation:

```
>>> I*I
-1
>>> solve( x**2 + 1 , x)
[I, -I]
```

The solutions are $x = i$ and $x = -i$, and indeed we can verify that $i^2 + 1 = 0$ and $(-i)^2 + 1 = 0$ since $i^2 = -1$.

The complex numbers \mathbb{C} are defined as $\{a + bi \mid a, b \in \mathbb{R}\}$. Complex numbers contain a real part and an imaginary part:

```
>>> z = 4 + 3*I
>>> z
4 + 3*I
>>> re(z)
4
>>> im(z)
3
```

The *polar* representation of a complex number is $z = |z| \angle \theta = |z| e^{i\theta}$. For a complex number $z = a + bi$, the quantity $|z| = \sqrt{a^2 + b^2}$ is known as the *absolute value* of z , and θ is its *phase* or its *argument*:

```
>>> Abs(z)
5
>>> arg(z)
atan(3/4)
```

The complex conjugate of $z = a + bi$ is the number $\bar{z} = a - bi$, which has the same absolute value as z but opposite phase:

```
>>> conjugate( z )
4 - 3*I
```

Complex conjugation is important for computing the absolute value of z ($|z| = \sqrt{z\bar{z}}$) and for division by z ($\frac{1}{z} = \frac{\bar{z}}{|z|^2}$).

Euler's formula

```
>>> from sympy import expand, rewrite
```

Euler's formula shows an important relation between the exponential function e^x and the trigonometric functions $\sin(x)$ and $\cos(x)$:

$$e^{ix} = \cos x + i \sin x.$$

To obtain this result in SymPy, you must specify that the number x is real and also tell `expand` that you're interested in complex expansions:

```
>>> x = symbols('x', real=True)
>>> expand(exp(I*x), complex=True)
cos(x) + I*sin(x)
>>> re( exp(I*x) )
cos(x)
>>> im( exp(I*x) )
sin(x)
```

Basically, $\cos(x)$ is the real part of e^{ix} , and $\sin(x)$ is the imaginary part of e^{ix} . Whaaat? I know it's weird, but weird things are bound to happen when you input imaginary numbers to functions.

Vectors

A vector $\vec{v} \in \mathbb{R}^n$ is an n -tuple of real numbers. For example, consider a vector that has three components:

$$\vec{v} = (v_1, v_2, v_3) \in \mathbb{R}^3.$$

To specify the vector \vec{v} , we specify the values for its three components v_1 , v_2 , and v_3 .

A matrix $A \in \mathbb{R}^{m \times n}$ is a rectangular array of real numbers with m rows and n columns. A vector is a special type of matrix; you can think of the vector $\vec{v} \in \mathbb{R}^n$ as a $1 \times n$ matrix. Because of this equivalence between vectors and matrices, in SymPy we use `Matrix` objects to represent vectors.

This is how we define vectors and compute their properties:

```
>>> u = Matrix([4,5,6])
>>> u
[4, 5, 6]                # 3-vector
>>> u[0]                  # 0-based indexing for entries
4
>>> u.norm()              # length of u
sqrt(77)
>>> uhat = u/u.norm()     # unit vector in same dir as u
>>> uhat
[4/sqrt(77), 5/sqrt(77), 6/sqrt(77)]
>>> uhat.norm()
1
```

Dot product

The dot product of the 3-vectors \vec{u} and \vec{v} can be defined two ways:

$$\vec{u} \cdot \vec{v} = \underbrace{u_x v_x + u_y v_y + u_z v_z}_{\text{algebraic def.}} = \underbrace{\|\vec{u}\| \|\vec{v}\| \cos(\varphi)}_{\text{geometric def.}} \in \mathbb{R},$$

where φ is the angle between the vectors \vec{u} and \vec{v} . In SymPy,

```
>>> u = Matrix([ 4,5,6])
>>> v = Matrix([-1,1,2])
>>> u.dot(v)
13
```

We can combine the algebraic and geometric formulas for the dot product to obtain the cosine of the angle between the vectors

$$\cos(\varphi) = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} = \frac{u_x v_x + u_y v_y + u_z v_z}{\|\vec{u}\| \|\vec{v}\|},$$

and use the `acos` function to find the angle measure:

```
>>> acos(u.dot(v)/(u.norm()*v.norm())).evalf()
0.921263115666387      # in radians = 52.76 degrees
```

Just by looking at the coordinates of the vectors \vec{u} and \vec{v} , it's difficult to determine their relative direction. Thanks to the dot product, however, we know the angle between the vectors is 52.76° , which means they *kind of* point in the same direction. Vectors that are at an angle $\varphi = 90^\circ$ are called *orthogonal*, meaning at right angles with each other. The dot product between two vectors is negative when the angle between them is $\varphi > 90^\circ$.

The notion of the “angle between vectors” applies more generally to vectors with any number of dimensions. The dot product for n -dimensional vectors is $\vec{u} \cdot \vec{v} = \sum_{i=1}^n u_i v_i$. This means we can talk about “the angle between” 1000-dimensional vectors. That’s pretty crazy if you think about it—there is no way we could possibly “visualize” 1000-dimensional vectors, yet given two such vectors we can tell if they point mostly in the same direction, in perpendicular directions, or mostly in opposite directions.

The dot product is a commutative operation $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$:

```
>>> u.dot(v) == v.dot(u)
True
```

Cross product

The *cross product*, denoted \times , takes two vectors as inputs and produces a vector as output. The cross products of individual basis elements are defined as follows:

$$\hat{i} \times \hat{j} = \hat{k}, \quad \hat{j} \times \hat{k} = \hat{i}, \quad \hat{k} \times \hat{i} = \hat{j}.$$

The cross product is defined by the following equation:

$$\vec{u} \times \vec{v} = (u_y v_z - u_z v_y, u_z v_x - u_x v_z, u_x v_y - u_y v_x).$$

Here’s how to compute the cross product of two vectors:


```
>>> u = Matrix([ 4,5,6])
>>> v = Matrix([-1,1,2])
>>> u.cross(v)
[4, -14, 9]
```

The vector $\vec{u} \times \vec{v}$ is orthogonal to both \vec{u} and \vec{v} . The norm of the cross product $\|\vec{u} \times \vec{v}\|$ is proportional to the lengths of the vectors and the sine of the angle between them:

```
(u.cross(v).norm()/(u.norm()*v.norm()))*n()
0.796366206088088      # = sin(0.921..)
```

The cross product is anticommutative, $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$:

```
>>> u.cross(v)
[4, -14, 9]
>>> v.cross(u)
[-4, 14, -9]
```

Watch out for this, because it's a new thing. The product of two numbers a and b is commutative: $ab = ba$. The dot product of two vectors \vec{u} and \vec{v} is commutative: $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$. However the cross product is not commutative: $\vec{u} \times \vec{v} \neq \vec{v} \times \vec{u}$, it is anticommutative: $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$.

Conclusion

I'll conclude with some words of warning about technological dependence. Computer technology is very powerful and everywhere around us, but we must not forget that computers are merely calculators that depend on human commands to direct them. This means it's important for you to learn how to do math "by hand" first, in order to know how to instruct computers to do the math for you.

I don't want you to use the tricks you learned in this tutorial to avoid learning how to do math and blindly rely on SymPy to do math calculations for you. That's not a good idea! I'll be very disappointed if you use SymPy to skip the "intellectual suffering" necessary to learn the new math concepts like numbers, equations, functions, etc. That's what math is all about—understand math concepts and the relationships between concepts. The part that is about rote memorized of math calculations procedures that you should have learned at school is not important at all. The tedious and repetitive math calculations is precisely what can be "outsourced" to SymPy.

To solve problems in math (or physics, chemistry, biology, etc.) the most important things are to: A) define the variables relevant for the problem, B) draw a diagram, and C) clearly set up the problem's equations in terms of the variables you defined. With these steps in

place, half the work of solving the problem is already done! Computers can't help with these important, initial modelling and problem-specific tasks—only humans are good at this stuff. Once you set up the problem (A, B, C), SymPy can help you breeze through any subsequent calculations that might be necessary to obtain the final answer.

Most of the big math and science discoveries were made using pen and paper, which shows that scribbling on paper is a useful as a tool for thinking. With what you learned about SymPy, you now have access to the combination of pencil and paper for thinking and SymPy for calculating. It's a very powerful combination! What is a real-world problem you'd like to solve? Try modelling the problem using math equations and see what happens. Go out there and do some science!

Links

[Installation instructions for jupyter notebook]

<https://jupyter.readthedocs.io/en/latest/install.html>

[The official SymPy tutorial]

<http://docs.sympy.org/latest/tutorial/intro.html>

[A list of SymPy gotchas]

<http://docs.sympy.org/dev/gotchas.html>

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WOJCIECH PIETRZAK, DEVELOPER



The author has more than 15 years of tutoring experience, a B.Eng. in electrical engineering, a M.Sc. in physics, and a Ph.D. in computer science from McGill University.

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