3.4 Analytical approximations

E3.30 Run the z-test for the mean of the kombucha volume measurements from Batch 05 to determine if it is a regular batch or not.

a) Formulate two competing hypotheses H_0 and H_A.
b) Calculate the sample mean $\overline{\mathbf{k}}_{05}$.
c) Find the sampling distribution of the mean $\overline{\mathbf{K}}_0$ under the population data model $K_0 \sim \mathcal{N}(\mu_{K_0} = 1000, \sigma_{K_0} = 10)$. Use the formula $\mathbf{se}_{\overline{\mathbf{k}}} = \frac{\sigma_{K_0}}{\sqrt{n}}$ to get the standard error of the mean.
d) Compute the two-sided p-value of the observed $\overline{\mathbf{k}}_{05}$ under $\overline{\mathbf{K}}_0$.
e) Make a decision about Batch 05 using the cutoff value $\alpha = 0.05$.

E3.31 Run a z-test for the mean μ of kombucha Batch 06. Compute a two-sided p-value using two methods:

a) Compute the z-statistic $z = \frac{\overline{\mathbf{k}}_{06} - \mu_{K_0}}{\mathbf{se}_{\overline{\mathbf{k}}}}$, where $\mathbf{se}_{\overline{\mathbf{k}}} = \frac{\sigma_{K_0}}{\sqrt{n}}$ is the standard error of the mean, then compute the p-value using the standard normal $Z_0 \sim \mathcal{N}(0, 1)$ as the reference distribution.
b) Compute the p-value directly from $\overline{\mathbf{k}}_{06}$ and the normal model for the sampling distribution of the mean $\overline{\mathbf{K}}_0 \sim \mathcal{N}(\mu_{K_0}, \mathbf{se}_{\overline{\mathbf{k}}})$.

Make a decision by comparing the p-value to the cutoff $\alpha = 0.05$.

E3.32 Run the two-sided t-test for the mean on Batch 05 of the kombucha dataset to determine if it is a regular batch or not.

a) Formulate the two competing hypotheses H_0 and H_A. Specify, in full detail, the data models under the two hypotheses and all assumptions about the model parameters you're making.
b) Calculate the mean $\overline{\mathbf{k}}_{05}$ and the standard deviation $s_{\mathbf{k}_{05}}$.
c) Calculate the estimated standard error using $\widehat{\mathbf{se}}_{\overline{\mathbf{k}}} = \frac{s_{\mathbf{k}_{05}}}{\sqrt{n}}$.
d) Compute the t-statistic using the formula $t = \frac{\overline{\mathbf{k}}_{05} - \mu_{K_0}}{\widehat{\mathbf{se}}_{\overline{\mathbf{k}}}}$.
e) Compute the two-sided p-value of the t-statistic under the standard t-distribution with $\nu = n - 1$ degrees of freedom.
f) Repeat the p-value calculation by computing the probability of observing a deviation of $\overline{\mathbf{k}}_{05}$ or larger under the model $T_{\overline{\mathbf{K}}_0} \sim \mathcal{T}(n-1, \mu_{K_0}, \widehat{\mathbf{se}}_{\overline{\mathbf{k}}})$ for the sampling distribution of the mean.
g) Make a decision about Batch 05 using the cutoff value $\alpha = 0.05$.

E3.33 Run a two-sided t-test for the mean of kombucha Batch 06.

a) Compute a two-sided p-value of $\overline{\mathbf{k}}_{06}$ under the sampling distribution of the mean $\overline{\mathbf{K}}_0 \approx T_{\overline{\mathbf{K}}_0} \sim \mathcal{T}(\nu, \mu_{K_0}, \frac{s_{\mathbf{k}_{06}}}{\sqrt{n}})$, with $\nu = n - 1$.
b) Make a decision by comparing the p-value to the cutoff $\alpha = 0.05$.
c) Compute a point estimate $\hat{\Delta}$ for the effect size $\Delta = \mu_{06} - \mu_{K_0}$ and a 90% confidence interval for the effect size $\mathbf{ci}_{\Delta,0.9}$.

Hint: Use the function `ci_mean` with the option `method="a"` to obtain a confidence interval for μ_{06} based on Student's t-distribution.

E3.34 Let's perform a χ^2-test for the variance of Batch 01 using the hypotheses $H_A : \sigma^2 > \sigma^2_{K_0}$ and $H_0 : \sigma^2 \leqslant \sigma^2_{K_0}$, where $\sigma^2_{K_0} = 100$.

a) Calculate the observed sample variance $s^2_{\mathbf{k}_{01}}$.
b) Compute the q-statistic using the formula $q = s^2_{\mathbf{k}_{01}} / \left(\frac{\sigma^2_{K_0}}{n-1}\right)$.
c) Compute the right-tailed p-value of the q-statistic under the standard χ^2-distribution with $\nu = n - 1$ degrees of freedom.
d) Make a decision about Batch 01 using the cutoff value $\alpha = 0.05$.

E3.35 Repeat the p-value calculation from E3.34 by computing the probability of the sample variance $s^2_{\mathbf{k}_{01}}$ or larger under the sampling distribution of the variance $S^2_{\mathbf{K}_0} \sim \chi^2\big(\texttt{df} = n-1,\ \texttt{scale} = \sigma^2_{K_0}/(n-1)\big)$.

E3.36 **Drug discovery**. The baseline model for the health indicator of people who have an ailment is $I_0 \sim \mathcal{N}(\mu_{I_0} = 100, \sigma_{I_0})$. A group of 10 patients are treated with a new drug, and their health indicator measurements after the treatment are recorded. The observed data is $\mathbf{i} = (101.9, 100.4, 99.9, 100.4, 101.1, 100.8, 100.8, 101.2, 99.8, 102.5)$. You want to know if the new drug has an effect on the health indicator so you run a hypothesis test based on the hypotheses $H_A : \mu \neq \mu_{I_0}$ and $H_0 : \mu = \mu_{I_0}$. Compute the p-value under the baseline model I_0 and make a decision using the cutoff $\alpha = 0.05$.

E3.37 Let's revisit exercise E3.28 from the end of Section 3.3. The five light deflection measurements from the 1919 eclipse were $\mathbf{d} = (0.63,\ 1.12,\ 1.61,\ 2.10,\ 2.59)$ arcseconds. The Newtonian prediction is a mean deflection of 0.87 arcseconds. Assume the population standard deviation σ_D is unknown so we must estimate it from the observed sample. Perform the one-sided t-test to compare the two hypotheses $H_0 : \mu_D \leqslant 0.87$ and $H_A : \mu_D > 0.87$. Calculate the p-value and state your conclusion at the $\alpha = 0.05$ significance level.

E3.38 Explain in your own words what is wrong with the following interpretations of p-values:

- **a)** A small p-value like $p = 0.02$ *proves* that H_A is true.
- **b)** The p-value = 0.02 obtained in a clinical trial of a new drug (compared to no treatment) *proves* that the new drug works.
- **c)** A large p-value like $p = 0.4$ *proves* that H_0 is true.
- **d)** The p-value 0.02 tells us the probability that H_0 is true is 2%.
- **e)** The p-value 0.02 tells us the probability that H_A is true is 98%.
- **f)** When the p-value is 0.02 and we decide to reject H_0, there is only a 2% probability that we made a *wrong decision*.
- **g)** The p-value of 0.02 tells us that there is a 98% probability that a *replication* of the experiment will also reject H_0.
- **h)** The tiny p-value 0.002 obtained in a clinical test of a new medication means the medication is *very* effective.

Hint: Recall the probability calculations used to compute p-values.