PROBLEMS AND EXERCISES
for the
No bullshit guide to math and physics

April 23, 2014
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Front matter

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The end of each section contains links to interesting webpages, animations, and further reading material. You can use these links as a starting point for further exploration.

The end of each chapter contains a series of exercises. Make sure you spend some quality time with them. You will learn a lot by solving exercises on your own.

Each chapter ends with a section of practice problems designed to test your understanding of the concepts developed in that chapter. Make sure you spend plenty of quality time with these problems to practice what you’ve learned. Figuring out how to use an equation on your own in the process of solving a problem is a much more valuable experience than simply memorizing the equation.

For optimal learning efficiency, I recommend that you spend as much time working through the practice problems as you will spend reading the lessons. Seeing which problems you find difficult to solve will tell you which sections of the chapter you need to revisit. An additional benefit of testing your skills on the practice problems is that you’ll be prepared in case a teacher ever tries to test you.

Throughout the book, I’ve included links to Internet resources like animations, demonstrations, and webpages with further reading material. Once you understand the basics, you’ll be able to understand a lot more Internet resources. The links provided are a starting point for further exploration.
Chapter 1

Math fundamentals

Chapter sections go here.

1.25 Math problems

We’ve now reached the first section of problems in this book. The purpose of the problems in this section is to make you practice the math fundamentals. In the real world, you’ll rarely need to solve equations by hand since you can use a computer for that purpose. However, knowing how to perform symbolic mathematical manipulations like factoring and completing the square will be useful in later chapters.

I have a special message to all readers who are learning math just for fun. You can either try the problems in this section or skip them. Since you have no exam on this material coming up, you could skip ahead to Chapter 2 without any immediate consequences. Still, you’ll be missing out if you don’t do the problems. I suggest that you do them, either later today or another time when you’re properly caffeinated, or you’ll likely forget most of what you’ve learned within a month. You’ll still remember the big ideas, but you’ll be fuzzy on the details. With math, it’s very much use it or lose it!

By solving some of the problems in this section, you’ll remember a lot more stuff. Your moments of suffering will be rewarded. But make sure you get away from the pixels while you’re solving problems. You don’t need fancy technology to do math; grab a pen and some paper from the printer and you’ll be fine. Do yourself a favour: put your phone in airplane-mode, close the lid of your laptop, and move away from desktop computers. Give yourself some time to think. Yes, I know you can look up the answer to any question in five seconds on the Internet, and you can use http://live.sympy.org to solve any math problem, but that is like outsourcing the thinking. Descartes,
Leibniz, and Riemann did most of their work with pen and paper and they did well. Spend some time with math the way the masters did.

P1.1 Solve for $x$ in the equation $x^2 - 9 = 7$.

P1.2 Solve for $x$ in the equation $\cos^{-1}(\frac{x}{4}) - x = \omega t$.

P1.3 Solve for $x$ in the equation $\frac{1}{x} = \frac{1}{3} + \frac{1}{2}$.

P1.4 Use a calculator to find the values of the following expressions:

(a) $\sqrt[3]{3}$  
(b) $2^{10}$  
(c) $7^{\frac{1}{3}} - 10$  
(d) $\frac{1}{2} \ln(e^{22})$

P1.5 Use substitution to solve for $x$ in the following equations:

(a) $x^6 - 4x^3 + 4 = 0$  
(b) $\frac{1}{2 - \sin x} = \sin x$

P1.6 Use the basic rules of algebra to simplify the following expressions:

(1) $\frac{ab}{a} b^2 c b^{-3}$  
(2) $\frac{abc}{bca}$  
(3) $\frac{27n^2}{\sqrt{9abba}}$

(4) $\frac{a(b + c) - ca}{b}$  
(5) $\frac{a}{c} \frac{b^4}{\sqrt{b}} a^2$  
(6) $(x + a)(x + b) - x(a + b)$

P1.7 Factor the following expressions as product of linear terms:

(1) $x^2 - 2x - 8$  
(2) $3x^3 - 27x$  
(3) $6x^2 + 11x - 21$

P1.8 A gold club and a golf ball cost $1.10 together. The golf club costs one dollar more than the ball. How much does the ball cost?

P1.9 An ancient artist drew scenes of hunting on the walls of a cave, including 43 figures of animals and people. There were 17 more figures of animals than people. How many figures of people did the artist draw?

P1.10 The father is 35 years old and his son is 5 years old. How many years later will the father’s age be four times the son’s age?

P1.11 A boy and a girl collected 120 nuts. The girl collected twice as many nuts as the boy. How many nuts did each collect?

P1.12 Alice is 5 years older than Bob. The sum of their ages is 25 years. How old is Alice?

P1.13 A publisher needs to bind 4500 books. One print shop can bind these books in 30 days, another shop can do it in 45 days. How many days are necessary to bind all the books if both shops work in parallel? Hint: Find the books-per-day rate of each shop.

P1.14 A plane leaves Vancouver travelling at 600 km/h toward Montreal. One hour later, a second plane leaves Vancouver heading for Montreal at 900 km/h. How long will it take for the second plane to overtake the first? Hint: The distance travelled is equal to the velocity multiplied by the time of traveled: $d = vt$.

P1.15 There are 26 sheep and 10 goats on a ship. How old is the captain?

P1.16 The golden ratio, denoted $\varphi$, is the positive solution to the quadratic equation $x^2 - x - 1 = 0$. Find the golden ratio.

P1.17 Solve for $x$ in the equation $\frac{1}{x} + \frac{2}{1 - x} = \frac{4}{x^2}$. Hint: Multiply both sides of the equation by $x^2(1 - x)$.

P1.18 Find the range of values of the parameter $m$ for which the equation $2x^2 - mx + m = 0$ has no real solutions. Hint: Use the quadratic formula.

P1.19 Use the properties of exponents and logarithms to simplify

(1) $\exp(x) e^{-x} \exp(z)$  
(2) $\left(\frac{xy}{x^2 y^3 z^{-4}}\right)^{-3}$  
(3) $(8x^6)^{-\frac{3}{2}}$

(4) $\log_4(\sqrt{2})$  
(5) $\log_{10}(0.001)$  
(6) $\ln(x^2 - 1) - \ln(x - 1)$

P1.20 Compute the following expressions involving fractions:

(1) $\frac{1}{2} + \frac{1}{4}$  
(2) $\frac{4}{7} - \frac{23}{5}$  
(3) $1\frac{3}{4} + 1\frac{31}{32}$

P1.21 Find the values of $x$ that satisfy the following inequalities:

(1) $2x - 5 > 3$  
(2) $5 \leq 3x - 4 \leq 14$  
(3) $2x^2 + x \geq 1$

P1.22 Two algorithms, P and Q, can be used to solve a certain problem. The running time of Algorithm P as a function of the size of the problem $n$ is described by the function $P(n) = 0.002n^2$. The running time of Algorithm Q is described by $Q(n) = 0.5n$. For small problems, Algorithm P runs faster. Starting from what $n$ will Algorithm Q be faster?

P1.23 Consider a right-angle triangle in which the shorter sides are 8 cm and 6 cm. What is the length of the triangle’s longest side?

P1.24 A television screen measures 26 inches on the diagonal. The screen height is 13 inches. How wide is the screen?

P1.25 A ladder of length 3.33 m is leaning against a wall and its foot is 1.44 m from the wall. What is the height $h$ where ladder touches the wall?
P1.26 **Kepler’s triangle** Consider a right-angle triangle in which the hypotenuse has length \( \varphi = \sqrt{\frac{5+1}{2}} \) (the golden ratio) and the adjacent side has length \( \sqrt{\varphi} \). What is the length of the opposite side?

P1.27 Find the lengths \( x, y, \) and \( z \) in the figure below.

P1.28 You’re observing a house from a blimp flying at an altitude of 2000 metres. From your point of view, the the house appears at an angle \( 24^\circ \) below the horizontal. What is the horizontal distance \( x \) between the blimp and the house?

P1.29 Given the angle and distance measurements labeled in Figure 1.29, calculate the distance \( d \) and the height of the mountain peak \( h \).

**Hint:** Use the definition of \( \tan \theta \) twice to obtain 2 equations in 2 unknowns.

P1.30 Find \( x \). Express your answer in terms of \( a, b, c \) and \( \theta \).

**Hint:** Use Pythagoras’ theorem twice and the tan function.

P1.31 Use the power-reduction trigonometric identities (page ??) to express \( \sin^2 \theta \cos^2 \theta \) in terms of \( \cos 4\theta \).

P1.32 An equilateral triangle is inscribed in a circle of radius 1. Find the side length \( a \) and the area of the inscribed triangle \( A_\triangle \).

**Hint:** Split the triangle into three sub-triangles.

P1.33 A circle of radius 1 is inscribed inside a regular octagon (an polygon with 8 sides of length \( b \)). Calculate the octagon’s perimeter and its area.
Hint: Split the octagon into 8 isosceles triangles.

**P1.34** Consider the obtuse triangle shown in Figure 1.2.

(a) Express $h$ in terms of $a$ and $\theta$.
(b) What is the area of this triangle?
(c) Express $c$ in terms of the variables $a$, $b$, and $\theta$

![Figure 1.2: A triangle with base $b$ and height $h$.](image)

Hint: You can use the cosine rule for part (c).

**P1.35** Find the length of side $c$ in the triangle:

![Diagram](image)

$C = 75^\circ$, $A = 41^\circ$, $b = 18$, $a = 10$

Hint: Use the sine rule.

**P1.36** Find the measure of the angle $B$ and deduce the measure of the angle $C$. Find the length of side $c$.

**P1.37** An observer on the ground measures an angle of inclination of $30^\circ$ to an approaching airplane, and 10 seconds later measures an angle of inclination of $55^\circ$. If the airplane is flying at a constant speed at an altitude of 2000 m in a straight line directly over the observer, find the speed of the airplane in kilometres per hour.

![Diagram](image)

Hint: The sum of the angle measures of any triangle is $180^\circ$.

**P1.38** Satoshi likes warm saké. He places 1 litre of water in a sauce pan with diameter 17 cm. How much will the height of the water level rise when Satoshi immerses a saké bottle with diameter 7.5 cm.

Hint: You’ll need the volume conversion ratio 1 litre = 1000 cm$^3$.

**P1.39** Find the length $x$ of the diagonal of the quadrilateral below.

![Diagram](image)

Hint: Use the law of cosines once to find $\alpha_1$ and $\alpha_2$, and again to find $x$.

**P1.40** Find the area of the shaded region.
Hint: Find the area of the outer circle, subtract the area of missing centre
disk, then divide by two.

**P1.41** In preparation for the shooting of a music video, you’re asked to
suspend a wrecking ball hanging from a circular pulley. The pulley has a
radius of 50 cm. The other lengths are indicated in the figure. What is the
total length of the rope required?

![Diagram of the pulley system](image)

Hint: The total length of rope consists of two straight parts and the curved
section that wraps around the pulley.

**P1.42** What is the measure of the angle $\theta$ in the figure below?

![Diagram of angle](image)

Hint: At the intersection of two lines, vertically opposite angles are equal.

**P1.43** A large circle of radius $R$ is surrounded by 12 smaller circles of
radius $r$. Find the ratio $\frac{R}{r}$ rounded to four decimals.

**P1.44** The area of a rectangular figure is 35 cm$^2$. If one side is 5 cm, how
long is the other side?

**P1.45** The length of a rectangle is $c + 2$ and its height is 5. What is the
area of the rectangle?

**P1.46** A box of facial tissues has dimensions 10.5 cm by 7 cm by 22.3 cm.
What is the volume of the box in litres?

**P1.47** A swimming pool has length $\ell = 20$ m, width $w = 10$ m, and depth
$d = 1.5$ m. Calculate the volume of water in the swimming pool in litres?

**P1.48** How many litres of water remain in a tank that is 12 m long, 6 m
wide, and 3 m high, if 30% of its capacity is spent?

**P1.49** A building has two water tanks, each with capacity 4000 L. One of
them is $\frac{1}{4}$-full and the other contains three times more water. How many
litres of water the building has now?

**P1.50** The rectangular lid of a box has length 40 cm and width 30 cm.
A rectangular hole with the area 500 cm$^2$ must be made in this lid so that
its sides were at equal distances from the sides of the lid. Which should be
the distances of sides of the hole from the sides of the lid?

**P1.51** A man sells firewood. To make standard portions, he uses a stan-
dard length of rope $\ell$ to surrounds a pack of logs. One day, a client asks
him to bring a double portion of firewood. What length of rope should he
use, assuming the packs of logs have roughly circular shape.
P1.52 How much pure water should be added to 10 litres of 60% solution of acid to make a 20% solution of acid?

P1.53 A tablet screen has a resolution of 768 pixels by 1024 pixels and the physical dimensions of the screen are 6 inches by 8 inches. One might conclude that the best choice of paper size for a PDF for such a screen would be 6 inches by 8 inches. At first I thought so too, but I forgot about the status bar, which is 20 pixels tall. The actual usable screen area is only 768 pixels by 1004 pixels. Assuming the width of the PDF is chosen to be 6 inches, what should be the height of the PDF so it fits perfectly in the content area of the tablet screen?

P1.54 Find the sum of the natural numbers 1 through 100. Hint: Imagine pairing the biggest number with the smallest number in the sum, the second biggest number with the second smallest number, etc.

P1.55 Solve the following system of 3 equations in 3 unknowns:
\[ \begin{align*}
1x + 2y + 3z &= 14, \\
2x + 5y + 6z &= 30, \\
-1x + 2y + 3z &= 12.
\end{align*} \]

P1.57 A hotel offers a 15% discount on rooms. Determine the original price of a room, if the discounted room price is $95.20.

P1.58 A set of kitchen tools retails for $450 but today there is on special offer for $360. Calculate the percentage of the discount.

P1.59 You take out a $5000 loan at nominal annual percentage rate (nAPR) of 12% and monthly compounding. How much money will you owe after 10 years?

P1.60 Plot the graphs of \( f(x) = 100e^{-x/2} \) and \( g(x) = 100(1 - e^{-x/2}) \) by evaluating the functions at different values of \( x \) from 0 to 11.

P1.61 Starting from an initial quantity \( Q_o \) of Exponentium at \( t = 0 \) s, the quantity \( Q \) of Exponentium as a function of time varies according to the expression \( Q(t) = Q_o e^{-t/\lambda} \), where \( \lambda = 5.0 \) and \( t \) is measured in seconds. Find the half-life of Exponentium, that is, the time taken for the quantity of Exponentium to reduce to half the initial quantity \( Q_o \).

P1.62 A hot body cools so that in each 24 min its temperature decreases by a factor of two. Deduce the time-constant and find how long it will take for the body to reach 1% of the original temperature. Hint: Temperature varies like \( T(t) = T_o e^{-t/\tau} \) and \( \tau \) is the time constant.

P1.63 A capacitor of capacitance \( C = 4.0 \times 10^{-6} \) farads, charged to an initial potential \( V_o = 20 \) volts, is discharging through a resistance of \( R = 10000 \) Ω (read Ohms). Find the potential \( V \) after 0.01 s and after 0.1 s, knowing the fall of potential follows the rule \( V(t) = V_o e^{-t/\tau} \).
Chapter 2

Introduction to physics

Chapter sections go here.

2.5 Kinematics problems

We spent an entire chapter learning about position, velocity, and acceleration equations used to describe the motion of objects. It’s now time to practice using these equations to solve problems.

Here are some general tips for solving kinematics problems. First try to figure out which equation you’ll need to solve the problem. There are just four of them:

\[
\begin{align*}
  x(t) &= x_i + v_i t + \frac{1}{2}at^2, \\
  v(t) &= v_i + at, \\
  a(t) &= a = \frac{F_{\text{net}}}{m}, \\ 
  v_f^2 &= v_i^2 + 2a\Delta x,
\end{align*}
\]

so it can’t be that hard. If you can’t figure it out, check the hint then try solving the problem. Always draw a diagram labelling all the variables that appear in your equations. This way you’ll always have a picture of what is going on. Check your answer against the answer provided on page 55. If you didn’t get the right answer, check your work and try again. Don’t look at the solution yet. Try to figure out the problem by revisiting the assumptions you made, the equations you wrote, and the steps you followed. Look at the solution only if you can’t figure out the problem after 10 minutes and you’re running out of ideas.

P2.1 Below is a velocity-vs-time graph of a moving particle. Is the particle gaining or losing speed? Does the graph describe uniformly accelerated motion (UAM) or not?
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Hint: Acceleration is the slope of the velocity graph.

P2.2 You’re running away from point A. At \( t = 2\,[\text{s}] \) you’re 3[\text{m}] away from A, at \( t = 4\,[\text{s}] \) you’re 8[\text{m}] away from A, and at \( t = 6\,[\text{s}] \) you’re 14[\text{m}] away from A. Are you running with uniform velocity (UVM)?
Hint: Calculate the velocity during each time interval.

P2.3 A car is moving on a straight road. Indicate whether the car’s speed is increasing or decreasing in the following cases:
1. Velocity is negative, acceleration is positive.
2. Velocity is negative, acceleration is negative.

Hint: Pay attention to the relative direction of acceleration to velocity.

P2.4 A body is pushed by a constant force \( F \), at time \( t = t_0 \) the force becomes zero. Determine when the particle is in uniformly accelerated motion (UAM) and when it is in uniform velocity motion (UVM).
Hint: Remember Newton’s 2nd law of motion.

P2.5 A car has the following acceleration-vs-time graph. The car starts from rest at \( t = 0\,[\text{s}] \). Find the velocity of the car at times A, B, C, D, and E.

Hint: The change in velocity is the area under the acceleration graph.

P2.6 The position of a rocket as a function of time is described by the equation \( x(t) = 3t^3 + 5t^2 - 3t + 5 \). Find the velocity and the acceleration of the rocket as functions of time.
Hint: Differentiate the function with respect to \( t \).

P2.7 You are on a mission to Jupiter where you design an experiment to measure the planet’s gravitational acceleration. In the experiment, you let go of a ball from a height of 4[m] and watch it fall to the ground. When the ball hits the ground, its speed is 14[m/s].
1. What is the gravitational acceleration on Jupiter?
2. Find the position of the ball as a function of time.

Hint: Use the fourth equation of motion.

P2.8 You’re pulling a 5[kg] cart in a straight path. The position of the cart as a function of time is \( x(t) = 6t^2 + 2t + 1\,[\text{m}] \).
1. Find the velocity and acceleration of the cart as functions of time.
2. Calculate the force you’re using to pull the cart.

Hint: Take the derivative of position with respect to time. Use Newton’s 2nd law \( F = ma \).

P2.9 A remote controlled car has a mass of 0.5[kg]. The electric engine pushes the car with a force of 1.0[N] starting from rest at point A.
1. Find the acceleration, velocity, and position of the car as functions of time, assuming \( x = 0 \) at point A.
2. Calculate the velocity of the car at \( t = 4\,[\text{s}] \).
3. What is the car’s velocity when it is 9[m] away from the point A?

Hint: Use Newton’s 2nd law and integration.

P2.10 Below is an acceleration-vs-time graph of a particle. At \( t = 0\,[\text{s}] \), the particle starts moving from rest at \( x = 0\,[\text{m}] \). The acceleration of the particle from \( t = 0\,[\text{s}] \) to \( t = 3\,[\text{s}] \) is given by \( a(t) = 3t\,[\text{m/s}^2] \). After \( t = 2\,[\text{s}] \), the acceleration is constant \( a = 6\,[\text{m/s}^2] \).
1. Find the velocity \( v(2) \) and position \( x(2) \) of the particle at \( t = 2\,[\text{s}] \).
2. Construct the functions of time that describe the acceleration, the velocity, and the position of the particle after \( t = 2\,[\text{s}] \).
3. How much time is needed for the particle to reach \( x = 49\,[\text{m}] \)?
4. At what distance from origin will the particle’s velocity reach 12[m/s]?
INTRODUCTION TO PHYSICS

Hint: Use integration to find the velocity and the position. The integral of \( f(t) = t^2 \) is \( F(t) = \frac{1}{3}t^3 \). Make sure that when \( t = 2\text{s} \), the functions \( v(t) \) and \( x(t) \) in Part 2 match your answer from Part 1.

P2.11 The graph below shows the position-vs-time graph of a squirrel running in a field where \( x \) is in metres and \( t \) is in seconds.

1. Calculate the velocity during the time intervals A to B, C to D, and E to F.
2. Indicate whether the squirrel is standing, moving forward (in the positive direction), or moving backward during the intervals: 0[s] to 2[s], 2[s] to 6[s], and 6[s] to 9[s].

Hint: Use integration to find the velocity and the position. The integral of \( f(t) = t^2 \) is \( F(t) = \frac{1}{3}t^3 \). Make sure that when \( t = 2\text{s} \), the functions \( v(t) \) and \( x(t) \) in Part 2 match your answer from Part 1.

P2.12 A car passes point A with velocity \( v_i \text{[m/s]} \) at \( t = 0\text{s} \), and has an acceleration of \(-2\text{[m/s}^2] \). The car comes to rest 9[m] away from point A.

1. What is \( v_i \)?

P2.13 Two dogs are running after a tennis ball. At \( t = 0\text{s} \) the first dog starts running from rest with an acceleration of 3[m/s]\(^2\), while the other dog is 4[m] ahead of the first dog at \( t = 0\text{s} \) running with velocity 3[m/s], and acceleration 1[m/s]\(^2\).

1. Construct the positions of the two dogs as functions of time.
2. At what time will the dogs meet?

Hint: Equate the positions of both dogs.

P2.14 A car has the following position function \( x(t) = 2t^2 + 5t + 7\text{[m]} \).

1. What are the position and velocity of the car at \( t = 0\text{s} \)?
2. Find the velocity and acceleration of the car as functions of time.
3. Find the position and the velocity of the car at \( t = 5\text{s} \)?

Hint: Use differentiation to find the velocity and acceleration functions.

P2.15 A car moving with initial velocity \( v_i \) applies the brakes. After 2[m] the car’s speed is 4[m/s] and 4[m] after applying the brakes the car comes to a stop. What is \( v_i \) and how much time was needed for the car to stop? Assume the car’s acceleration (deceleration) is constant starting from the point the brakes are applied until it stops.

Hint: Use the fourth equation of motion. Find the position function.

2. What is the position of the car as a function of time?

Hint: Use the fourth equation of motion.

P2.16 A car passes point A with velocity \( v_i \text{[m/s]} \) at \( t = 0\text{s} \), and has an acceleration of \(-2\text{[m/s}^2] \). The car comes to rest 9[m] away from point A.

1. What is \( v_i \)?
Chapter 3

Vectors

Chapter sections go here.

3.6 Vectors problems

You learned a bunch of vector formulas and you saw some vector diagrams, but did you really learn how to solve problems with vectors? There is only one way to find out: test yourself by solving problems.

I’ve said it before and I don’t want to repeat myself too much, but it’s worth saying again: the more problems you solve, the better you will understand the material. It’s now time for you to try the following vector problems to make sure you’re on top of things.

P3.1 Express the following vectors in length-and-direction notation:

(a) \( \vec{u}_1 = (0, 5) \)  
(b) \( \vec{u}_2 = (1, 2) \)  
(c) \( \vec{u}_3 = (-1, -2) \)

P3.2 Express the following vectors as components:

(a) \( \vec{v}_1 = 20\angle30^\circ \)  
(b) \( \vec{v}_2 = 10\angle-90^\circ \)  
(c) \( \vec{v}_3 = 5\angle150^\circ \)

P3.3 Express the following vectors in terms of unit vectors \( \hat{i}, \hat{j}, \) and \( \hat{k} \):

(a) \( \vec{w}_1 = 10\angle25^\circ \)  
(b) \( \vec{w}_2 = 7\angle-90^\circ \)  
(c) \( \vec{w}_3 = (3, -2, 3) \)

P3.4 Given the vectors \( \vec{v}_1 = (1, 1), \vec{v}_2 = (2, 3), \) and \( \vec{v}_3 = 5\angle30^\circ, \) calculate the following expressions:

(a) \( \vec{v}_1 + \vec{v}_2 \)  
(b) \( \vec{v}_2 - 2\vec{v}_1 \)  
(c) \( \vec{v}_1 + \vec{v}_2 + \vec{v}_3 \)

P3.5 Starting from the point \( P = (2, 6) \), the three displacement vectors shown in Figure 3.1 are applied to obtain the point \( Q \). What are the coordinates of the point \( Q \)?
To bring an idea from the imaginary into the real, you must work on it. We’ll model the work done on the project as a multiplication by the complex number $e^{-i\alpha h}$, where $h$ is the number of hours of work and $\alpha$ is a constant that depends on the project. After $h$ hours of work, the initial state of the project is transformed as follows: $p_f = e^{-i\alpha h} p_i$. Working on the project for one hour “rotates” its state by $-\alpha$[rad], making it less imaginary and more real.

If you start from an idea $p_0 = 100i$ and the cumulative number of hours invested after $t$ weeks of working on the project is $h(t) = 0.2t^2$, how long will it take for the project to become 100% real? Assume $\alpha = 2.904 \times 10^{-3}$. Hint: A project is 100% real if $\text{Re}\{p\} = p$.

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**Figure 3.1:** A point $P$ is displaced by three vectors to obtain point $Q$.

**P3.6** Given the vectors $\vec{u} = (1, 1, 1), \vec{v} = (2, 3, 1)$, and $\vec{w} = (-1, -1, 2)$, compute the following products:

1. $\vec{u} \cdot \vec{v}$
2. $\vec{u} \cdot \vec{w}$
3. $\vec{v} \cdot \vec{w}$
4. $\vec{u} \times \vec{v}$
5. $\vec{u} \times \vec{w}$
6. $\vec{v} \times \vec{w}$

**P3.7** Find a unit-length vector that is perpendicular to both $\vec{u} = (1, 0, 1)$ and $\vec{v} = (1, 2, 0)$.

Hint: Use the cross product.

**P3.8** Find a vector that is orthogonal to both $\vec{u}_1 = (1, 0, 1)$ and $\vec{u}_2 = (1, 3, 0)$ and whose dot product with the vector $\vec{v} = (1, 1, 0)$ is equal to 8.

**P3.9** Compute the following expressions:

a) $\sqrt{-4}$

b) $\frac{2 + 3i}{2 + 2i}$

c) $e^{3i}(2 + i)e^{-3i}$

**P3.10** Solve for $x \in \mathbb{C}$ in the following equations:

a) $x^2 = -4$

b) $\sqrt{x} = 4i$

c) $x^2 + 2x + 2 = 0$

(d) $x^4 + 4x^2 + 3 = 0$

Hint: To solve (d), use the substitution $u = x^2$.

**P3.11** Given the numbers $z_1 = 2 + i, z_2 = 2 - i,$ and $z_3 = -1 - i$, compute

a) $|z_1|$

b) $\frac{z_1}{z_3}$

c) $z_1 z_2 z_3$

**P3.12** A real business is a business that is profitable. An imaginary business is an idea that is just turning around in your head. We can model the real-imaginary nature of a business project by representing the project state as a complex number $p \in \mathbb{C}$. For example, a business idea is described by the state $p_0 = 100i$. In other words, it is 100% imaginary.
Chapter 4

Mechanics

Chapter sections go here.

4.5 Momentum

Exercises

E4.1 A sticky ball of mass 3\(\text{g}\) and velocity 20\(\text{m/s}\) collides with a stationary ball of mass 5\(\text{g}\). The balls stick together. What is their velocity after the collision?

Hint: Use conservation of momentum \(\vec{p}_{1,\text{in}} + \vec{p}_{2,\text{in}} = \vec{p}_{\text{out}}\).

More chapter sections go here.

4.11 Mechanics problems

It’s now time for you to verify experimentally how well you’ve understood the material from this chapter. Try solving the physics problems presented in this section. Don’t be discouraged if you find some of the problems difficult—they are meant to be challenging in order to force you to think hard and reinforce the connections between the concepts in your head.

When solving physics problems, I recommend that you follow this 5-step procedure:

1. Figure out what type of problem you are dealing with. Is it a kinematics problem? A momentum problem? An energy problem? A problem about angular motion?

2. Draw a diagram that describes the physical situation. If the problem involves vectors, draw a coordinate system. Label the known and the unknown quantities in the diagram.
3. Write down the physics **formulas** that are usually used for the type of problem you're solving. You can copy the necessary formulas from the table on page ??.

4. Substitute the **known quantities** into the equations and determine which unknown(s) you need to find. Visualize the steps you'll take to solve for the unknown(s).

5. Do the math.

Note that math appears only in the last step. If you want to solve a physics problem and the first thing you do is manipulate equations and numbers then you're shooting yourself in the foot. Physics is not about solving equations but about thinking abstractly about the "moving parts" in the problem: positions, velocities, energies, etc. As far as I'm concerned, if you do Steps 1, 2, 3, and 4 correctly and make a mistake in Step 5, you're good in my books. Manipulating math equations fluently and errorlessly is a skill that takes time to hone. If you're still new to the techniques covered in Chapters 1–3, it’s normal to make mistakes. Don’t worry about it, just practice.

Make sure you attempt each of the exercises on your own before looking at the answers and the solutions. If you want to practice Step 1 of the “solving physics problems” procedure, don’t look at the hints. The first step is very strategic so you need to practice it. The problems are intentionally presented out of order, to force you to think about Step 1. Knowing what type of problem you are dealing with is the part that most closely resembles what physics research is like. Given a physics question, physicists try to visualize the situation, label the variables of the problem, and then ask “What can I use here?” Remember I said that using physics equations is similar to playing with **legos**? You must find which physics equation (or principle) “fits” the problem. Once you know the type of problem you are dealing with, writing down the equations and doing the math are comparatively easier tasks. The cool part about learning physics in the “controlled environment” of this problem set is that one of the equations you learned is guaranteed to work.¹

**P4.1** You throw a water balloon from ground level with initial velocity \( \vec{v}_i \) at an angle \( \theta \) above the horizon.

1. Find \( v_y(t) \), the vertical velocity of the balloon as a function of time: (a) when the \( y \)-axis points up and (b) when the \( y \)-axis points down.
2. A cat starts running away from you just as you throw the balloon. If the cat’s horizontal velocity \( v_{\text{cat}} \) is equal to \( v_{ix} \) of the balloon, will the cat get splashed by the balloon?

¹In research, it’s not always like that; sometimes there is no “known” strategy to follow and you must come up with a new approach to solve the problem.

**P4.2** The four vectors in the diagram below have the same magnitude. Place \( \vec{F}_4 \) properly (you can change its direction) to achieve the following cases: (1) \( \vec{a}_{\text{block}} = 0 \), (2) \( \vec{a}_{\text{block}} \) has a upward component, and (3) \( \vec{a}_{\text{block}} \) has a single component directed to the left.

**P4.3** Two particles: the first has mass \( m \) and speed \( 2v \), the second has mass \( 2m \) and speed \( v \). Compare the magnitudes of their momenta and their kinetic energies.

**P4.4** A space station has two identical compartments A and B and is moving with velocity \( \vec{v} \) in space. An explosive charge separates the two compartments and they continue with velocities \( \vec{v}_A \) and \( \vec{v}_B \). Find \( \vec{v}_B \) in the following three cases: (1) \( \vec{v}_A = \vec{v} \), (2) \( \vec{v}_A = -\vec{v} \), and (3) \( \vec{v}_A = 0 \).

**P4.5** A 10[cm] spring is suspended vertically and a mass \( m \) hangs from it. What are the types of energies in the system when the mass \( m \) is in positions 1 and 2 below? Measure \( U_g \) relative to the height \( y = 0 \).
P4.6 You throw a ball from the ground vertically with a speed $v$ and measure its speed when it comes back to the ground. You first carry out this experiment on Earth then repeat it on the Moon. Where does the ball have a greater speed as it hits the ground, on Earth or on the Moon?
Hint: Use conservation of energy.

P4.7 In the previous problem, assume that there is a pit that allows the ball to fall 10[m] below the level from which it was thrown. Will the ball have a greater speed on Earth or on the Moon when it hits the bottom of the 10[m]-deep pit?
Hint: Use conservation of energy.

P4.8 Two balls of mass $m$ are thrown from the top of a building with equal velocity $v$ as shown in the diagram. Which ball has the greatest speed at the moment it hits the ground (ignore air resistance)?

![Diagram of two balls being thrown from a building]

Hint: Use conservation of energy.

P4.9 A circular loop is placed vertically with a car rolling inside it. If the car passes the bottom and top of the loop at the same speed, where will the normal force exerted by the loop on the car be greater?
Hint: The car requires a centripetal force to maintain its circular path.

P4.10 A rod of mass $m$ is rotating horizontally about one of its ends. An additional mass $M$ is attached at the other end. Assume the system rotates at a constant angular velocity $\omega$. What is the torque on the mass $M$? If the mass $M$ is detached from the rod without any intervention of an external force, what will be the new angular velocity of the rod?
Hint: Use $T = I \alpha$. Use conservation of angular momentum.

P4.11 Three pendulum clocks are made using strings of the same length, but each clock has a different swinging amplitude: $\theta_{\text{max},1} = 5^\circ$, $\theta_{\text{max},2} = 7.5^\circ$, and $\theta_{\text{max},3} = 10^\circ$. Will the clocks measure time consistently?
Hint: Remember that $T = 2\pi \sqrt{\frac{L}{g}}$.

P4.12 Three pendulum clocks are made using pendulums of the same mass but different lengths: $\ell_1 = 50$[cm], $\ell_2 = 75$[cm], and $\ell_3 = 100$[cm]. Find the period $T$ of each pendulum on Earth?

P4.13 You start a pendulum from an initial angle $\theta_{\text{max}}$ on Earth then on the Moon. The pendulum consists of a mass $m$ suspended on a string of length $\ell$. Will the mass have greater speed as it passes through $\theta = 0$ on Earth or on the Moon?
Hint: Use conservation of energy.

P4.14 A diver jumps into a swimming pool from a platform that is 10 m above the water and does several flips while in midair. What types of energy exist while the diver is in the air? What kinds of momenta are there?
Hint: Remember the diver is rotating in midair.

P4.15 A pendulum is made from a mass $m$ suspended on a long string. The string is being wound up by motor at a certain rate thus making the pendulum shorter. What effect does this have on the pendulum’s period?
Hint: Remember that $T = 2\pi \sqrt{\frac{L}{g}}$.

P4.16 You’re pushing a 15[kg] box of stuff. The kinetic coefficient of friction between the box and the floor is $\mu_k = 0.032$. How much force should you apply on the box to keep it moving with constant velocity?
Hint: Constant velocity implies zero acceleration.

P4.17 You and your friend are playing frisbee. Your friend throws the 175[g] frisbee at you with a speed $v = 15$[m/s]. The frisbee flies horizontally with a constant speed until you catch it (ignore air resistance and gravity). How much energy does the frisbee lose when you stop it with your hand?

P4.18 Two football players collide head on then fall on the ground at the same place without moving further. Their masses are $m_1 = 90$[kg] and $m_2 = 75$[kg]. Their velocities before the impact are $v_1 = 4.5$[m/s] and $v_2$. Consider the players to be point masses on a frictionless field. (1) What is $v_2$? (2) Was this an elastic collision?
Hint: Use conservation of momentum.

P4.19 The sliding blackboard in your classroom has two panels of mass $m = 20$[kg] balanced on a pulley. See Figure 4.1. Each of the blackboard panels is 1.5[m] in height. You notice that your professor switches the boards by pulling board A down with a constant force $F$ for 0.5[m] until it reaches a speed of 1[m/s], then he allows the panel to slide down freely for another 0.5[m]. In the last 0.5[m] he exerts an upward force on the panel to decrease its speed to zero by the end of the motion. In the end, the upper board A is lowered by 1.5[m] and board B is raised by 1.5[m]. Assume frictionless, massless pulley and ropes.

1. Draw a force diagram of the two boards during each of the three stages of the motion.
2. Calculate the acceleration of board A during each of the three stages.
3. Calculate the force exerted by the professor on board A during the first stage.
Hint: Pay attention to tension force that connects the two panels and use Newton’s 2nd law of motion.
Define $W_a[J]$ to be the work required to get an object of mass $m[kg]$ moving at speed $v[m/s]$ by starting from rest and pushing the object in a straight line on a frictionless surface. Suppose you now do double the amount of work $W = 2W_a[J]$ to compress a spring from its normal length to a certain shorter length. You then fix one side of the compressed spring, put the same object in front of the spring and release the spring. What is the velocity of the object after it gets pushed by the spring?

Hint: Use conservation of energy.

The moment of inertia of a door is $I_{\text{door}} = 11.4[kg\cdot m^2]$. You want to accelerate it from rest to an angular velocity of $\omega = 1.3[rad/s]$ in 3 seconds using uniform angular acceleration.

1. Calculate the torque you need to apply on the door.
2. How much work will you do during this process?

Hint: Use kinematics and $T = I\alpha$.

A spring with stiffness $k = 115[N/m]$ is compressed by $\Delta x = 40[cm]$ then released to push a ball of radius $R = 10[cm]$ placed in front of it. When the ball leaves the spring it rolls without skidding at an angular velocity of $\omega = 30[rad/s]$. What is the mass of the ball?

Hint: Use the angular velocity and the radius to find the linear velocity of the ball. Recall that $I_{\text{ball}} = \frac{2}{5}mR^2$.

Two blocks are on top of each other. The mass of the upper block is $m_1 = 0.25[kg]$ and the lower block’s mass is $m_2 = 1.00[kg]$. You push the lower block with a constant force $F = 3.0[N]$. Find the minimum coefficient of static friction between the blocks for which the upper block will not slip.

Figure 4.1: The sliding blackboard mechanism discussed in P4.19.

Figure 4.2: Hitting a moving target.

Figure 4.3: The three-mass system analyzed in P4.25.
P4.26 A ball is fired with $v_{1i} = 10\text{[m/s]}$ at an angle $\theta = 30^\circ$ with the horizontal. You need to fire a second ball vertically from the same height such that it hits the first ball as it reaches its maximum height.

1. Find the horizontal distance $d$ from the first ball’s firing position from where you should fire the second ball.

2. Find the initial velocity required for the second ball.

Hint: Find the maximum height and the half-range of the first ball.

P4.27 A 14000[kg] F-16 fighter jet is in a dogfight. The pilot needs to make a turn with a radius of 5[km] while maintaining a speed of 605.5[km/h]. The plane executes a banked turn at an angle $\theta = 30^\circ$ and follows a horizontal circular path.

1. How much lift does the pilot need to perform this maneuver? The lift force is perpendicular to the wings and the fuselage of the plane.

2. Will the altitude of the fighter change during this maneuver?

Hint: Find the lift force needed to produce the centripetal acceleration.

P4.28 You’re in a subway car moving at $v = 12.5[\text{m/s}]$ when you drop a water bottle on the floor. The bottle comes to rest with respect to the subway car. The subway then starts braking and comes to a stop. The bottle starts rolling forward without slipping on the floor of the subway car. Find the linear velocity of the bottle as it rolls forward. The bottle’s moment of inertia is $I = \frac{1}{2}mr^2$ and its mass is $m$.

Hint: The kinetic energy of the bottle is conserved.

P4.29 You’re playing with two hockey pucks on a pool table as shown in Figure 4.4. The coefficient of friction between the pucks and the table is $\mu_k$. Puck 2 is at rest before the collision and at a distance $d$ from the corner pocket. Puck 1 hits Puck 2 ($m_1 = m_2$) with velocity $v_i$. After the collision, Puck 1 has velocity $v_1$ and Puck 2 has velocity $v_2$.

1. What is the minimum $v_i$ in terms of the variables provided such that Puck 2 enters the pocket?

2. Calculate $v_i$, $v_1$, and $v_2$ if $\mu_k = 0.273$ and $d = 0.70[\text{m}]$.

Hint: Use an energy calculation, then momentum, then energy again.

P4.30 A car accelerates at $a = 4[\text{m/s}^2]$. Each tire has radius $r = 30[\text{cm}]$ and moment of inertia $I = 0.27[\text{kg m}^2]$. The car doesn’t skid.

1. What is the torque applied on each tire?

2. If $\theta_i = 0$ and $\omega_i = 3[\text{rad/s}]$, how many revolutions did the tire complete from $t = 0[\text{s}]$ until $t = 4[\text{s}]$.

Hint: Use the relation between $a$ and $\alpha$.

P4.31 A solid cylinder and a hollow cylinder of identical mass are placed side by side on an incline. If both cylinders are released from rest and start rolling, which cylinder will reach the bottom of the incline first?

Hint: Think about $T = I\alpha$.

P4.32 A pendulum of mass $M$ is released from rest when the string is perfectly horizontal and swings down to hit the box of mass $m$. The pendulum string is vertical when the collision occurs and the pendulum stops after it hits the box (it doesn’t bounce back). If the box slides a distance $d$ along the horizontal surface after the collision, what is the coefficient of kinetic friction $\mu_k$ between the box and the surface. State your answer in terms of the quantities $M$, $m$, $L$, $d$, and $g$.

Hint: Use an energy calculation, then momentum, then energy again.

P4.33 The gravitational acceleration on Earth is not the same everywhere. The weakest gravitational acceleration is at the top of the Nevado Huascaran summit in Peru $g_{\text{min}} = 9.76[\text{m/s}^2]$. The strongest $g$ is on the
North Pole where it is $g_{\text{max}} = 9.83\text{[m/s}^2]\text{]. Ignoring the effects of air frictions, how much further will a football travel if kicked with initial velocity $30\angle45^\circ\text{[m/s]}$ on the top of Nevado Huascarán compared to the North Pole? Hint: This problem requires two range calculations for projectile motion.

P4.34 A ball is thrown from ground level with upward initial velocity $20\text{[m/s]}$. How long will the ball be in the air before it returns to the ground?

P4.35 Given $a(t) = 4\text{[m/s}^2]\text{], } v_i = 10\text{[m/s]}\text{, } x_i = 20\text{[m]}\text{, find } x(t)\text{, the position as a function of time } t\text{[s]}\text{.}$

P4.36 A disk is rotating with angular velocity $\omega = 5\text{[rad/s]}\text{.}$ A slug is sliding along the surface of the disk in the radial direction. The slug starts from the disk’s centre and is moving outward. If the coefficient of friction between the slug and the disk is $\mu_s = 0.4\text{,}$ how far can the slug slide before it flies off the surface of the disk?

P4.37 You have loaded a fridge into an elevator. Due to the static force of friction, the refrigerator needs a strong push to start it sliding across the elevator floor. From smallest to largest, rank the magnitude of the static force of friction in these three situations: a stationary elevator, an upward accelerating elevator, and a downward accelerating elevator.

P4.38 Three coins are placed on a turntable. One coin is placed $5\text{[cm]}$ from the turntable’s centre, another is placed $10\text{[cm]}$ from the centre, and the third is placed $15\text{[cm]}$ from the centre. The turntable is powered on and begins to spin. Initially, due to static friction, the coins move together with the turntable as it starts rotating. The angular speed $\omega$ of the turntable then increases slowly. Assuming all the coins have the same coefficient of friction with the turntable surface, which coin begins to slide first? Hint: This is a circular motion question.

P4.39 Two identical pulleys with the same moment of inertia but different radii have strings wound around them. The first pulley has radius $R$, while the second pulley has a smaller radius $r < R$. The same force $F$ is applied to pull on the string and rotate the pulleys. After a fixed time $t$, which pulley has the faster rotational speed? Which pulley has the greater rotational kinetic energy? Hint: This is an angular motion question.

The following exercises require a mix of techniques from different sections.

P4.40 The disk brake pads on your new bicycle squeeze the brake disks with a force of $5000\text{[N]}$ from each side. There is one brake pad on each tire. The coefficient of friction between the brake pads and brake disks is $\mu_k = 0.3\text{.}$ The brake disks have radius $r = 6\text{[cm]}$ and the bike’s tires have radius $R = 20\text{[cm]}\text{.}$
1. What is the total force of friction exerted by each brake?
2. What is the torque exerted by each brake?
3. Suppose you’re moving at $10\text{[m/s]}$ when you apply broth brakes. The combined mass of you and your bicycle is $100\text{[kg]}\text{.}$ How many times will the wheels turn before the bike stops?
4. What will be the braking distance?

P4.41 Tarzan A half-naked dude swings from a long rope attached to the ceiling. The rope has length $6\text{[m]}\text{.}$ The dude swings from an initial angle of $-50^\circ$ ($50^\circ$ to the left of the rope’s vertical line) all the way to the angle $+10^\circ$, at which point he lets go of the rope. How far will Tarzan fall, as measured from the centre position of the swing motion? Find $x_f = 6\sin(10) + d$ where $d$ is the distance travelled by Tarzan after he lets go. Hint: This is an energy problem followed by a projectile motion problem.

P4.42 A disgruntled airport employee decides to vandalize a moving walkway by suspending a leaking-paint-bucket pendulum above it. The pendulum is composed of a long cable (considered massless) and a paint bucket with a hole in the bottom. The pendulum’s oscillations are small, and transverse to the direction of the walkway’s motion. Find the equation $y(x)$ of the pattern of paint that forms on the moving walkway in terms of the pendulum’s maximum angular displacement $\theta_{\text{max}}\text{,}$ its length $\ell\text{,}$ and the speed of the walkway $v\text{.}$ Assume $x$ measures distance along the walkway and $y$ denotes the transversal displacement from the centre of the walkway. Hint: This is a simple harmonic motion question involving a pendulum.

Links
- A wikibook of physics exercises with solutions 
- en.wikibooks.org/wiki/Physics_Exercises
- Lots of interesting worked examples
- farside.ph.utexas.edu/teaching/301/lectures/lectures.html
- Interactive exercises from the MIT mechanics course on edX
- www.edx.org/course/mitx/mitx-8-01x-classical-mechanics-853
Chapter 5

Calculus

Chapter sections go here.

5.4 Limits

Exercises

E5.1 Calculate the following limit expressions:

(a) \( \lim_{x \to \infty} \frac{1}{x + 4} \) \hspace{1cm} (b) \( \lim_{x \to \infty} \frac{2x + 2}{x + 4} \) \hspace{1cm} (c) \( \lim_{x \to \infty} \frac{x^2 + 2}{x + 4} \)

Hint: Use L’Hospital’s rule for (b) and (c).

5.16 Applications of integration

Exercises

E5.2 Calculate the volume of a cone with radius \( R \) and height \( h \) that is generated by the revolution around the \( x \)-axis of the region bounded by the curve \( y = R - \frac{R}{h} x \) and the lines \( y = 0 \) and \( x = 0 \).

E5.3 Calculate the volume of the solid of revolution generated by revolving the region bounded by the curve \( y = x^2 \) and the lines \( x = 0 \), \( x = 1 \), and \( y = 0 \) around the \( x \)-axis.

E5.4 Calculate the volume of the solid of revolution generated by revolving the region bounded by the curves \( y = x^2 \) and \( y = x^3 \) and the lines \( x = 0 \) and \( x = 1 \) around the \( x \)-axis.

E5.5 Find the volume of a vertical cone with radius \( R \) and height \( h \) formed by the revolution of the region bounded by the curves \( y = \).
$h = \frac{h}{R}x, \ y = 0 \ and \ x = 0, \ around \ the \ y-axis. \ Use \ the \ cylindrical \ shell \ method.

### 5.21 Calculus problems

In this chapter we learned about derivatives and integrals, which are mathematical operations that relate to the slope of a function and the area under the graph of a function. We also learned about limits, sequences, and series. It’s now time to see how much you’ve really learned by trying to solve some calculus problems.

Calculus hasn’t changed much in the last hundred years. A testament to this is the fact that many of the problems presented here were adapted from the book “Calculus Made Easy” by Silvanus Thompson originally published\(^1\) in 1910. These problems are equally pertinent and interesting today as they were one hundred years ago.

As much as calculus is about understanding things conceptually and seeing the big picture (abstraction), calculus is also about practice. There are more than 100 problems to solve in this section. The goal is to turn differentiation and integration into routine operations that you can carry out without stressing out. You should vanquish as many problems as you need to feel comfortable with the procedures of calculus.

Okay, enough prep talk. Let’s get to the problems.

#### Limits problems

**P5.1** Use the graph of the function $f(x)$ shown in Figure 5.1 to calculate the following limit expressions:

1. $\lim_{x \to -5} f(x)$
2. $\lim_{x \to -5^+} f(x)$
3. $\lim_{x \to -5^-} f(x)$
4. $\lim_{x \to 2^-} f(x)$
5. $\lim_{x \to 2^+} f(x)$
6. $\lim_{x \to 2^-} f(x)$
7. $\lim_{x \to 5^-} f(x)$
8. $\lim_{x \to 5^+} f(x)$
9. $\lim_{x \to 5^-} f(x)$
10. Is the function $f(x)$ continuous at $x = 5$?
11. What are the intervals where the function $f(x)$ is continuous?

**P5.2** Find the value of the following limit expressions:

(a) $\lim_{x \to -3} 4$
(b) $\lim_{x \to -3} 2x$
(c) $\lim_{x \to -3} x^2 - 2x + 2$

**P5.3** Prove the following limit statement by constructing an $\epsilon, \delta$-proof:

$\lim_{x \to 5} 3x = 15.$

Recall the $\epsilon, \delta$-game: one player (the sceptic) specifies the required precision $\epsilon > 0$, and the other player (the prover) must find a value $\delta > 0$ such that $|3x - 15| \leq \epsilon$ for all $x$ in the interval $(5 - \delta, 5 + \delta)$. Hint: Choose $\delta$ to be a multiple of $\epsilon$.

**P5.4** Calculate the limit if it exists or explain why it doesn’t.

(a) $\lim_{x \to -\infty} \sin(x)$
(b) $\lim_{x \to 0^+} \sin(x)$
(c) $\lim_{x \to -\infty} \sin(x)$
(d) $\lim_{x \to 0^+} \sin(\frac{1}{x})$
(e) $\lim_{x \to 0^+} \sin(\frac{1}{x})$
(f) $\lim_{x \to -\infty} \sin(\frac{1}{x})$
(g) $\lim_{x \to 0^+} x\sin(\frac{1}{x})$
(h) $\lim_{x \to 0^+} x\sin(\frac{1}{x})$
(i) $\lim_{x \to -\infty} x\sin(\frac{1}{x})$

Hint: Using the substitution $y = \frac{1}{x}$, you can rewrite $\lim_{x \to 0^+} \sin(\frac{1}{x}) = \lim_{y \to \infty} \sin(y)$.

**P5.5** Calculate the following limit expressions:

(a) $\lim_{x \to 1} \frac{x^2 + 5x + 6}{x - 1}$
(b) $\lim_{x \to 1} \frac{x^2 + x - 2}{x - 1}$
(c) $\lim_{x \to a} \frac{x^2 - a^2}{x - a}$

**P5.6** Use a calculator to verify numerically the limits (1) through (6):

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\(^1\)Full text is available here: [gutenberg.org/ebooks/33283](http://gutenberg.org/ebooks/33283) (public domain).
P5.8

(1) \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e \\
(2) \lim_{x \to 0} \frac{x}{e^x} = 0 \\
(3) \lim_{x \to 0} \frac{\ln x}{x} = 0 \\
(4) \lim_{x \to 0} (1 + e^{1/x}) = e \\
(5) \lim_{x \to 0} \frac{\sin x}{x} = 1 \\
(6) \lim_{x \to 0} \cos x = 1 \\
(7) \text{Prove (5) and (6) by expanding \( \sin \) and \( \cos \) as Maclaurin series.}

Stage 1 cleared! I hope working through these problems helped you feel more confident about limits. If you liked the \( \epsilon, \delta \)-type of argument used in P5.3, you should look into learning analysis. Analysis is like calculus but with proper proofs from first principles.

Derivatives problems

P5.7 Find the derivative with respect to \( x \) of the functions:

(1) \( y = x^{13} \) 
(2) \( y = x^{-\frac{3}{2}} \) 
(3) \( y = x^{2a} \)

(4) \( u = t^{2.4} \) 
(5) \( z = \sqrt[3]{u} \) 
(6) \( y = \sqrt[3]{x-3} \)

(7) \( u = \sqrt[n]{x^n} \) 
(8) \( y = 2x^n \) 
(9) \( y = \sqrt[3]{x^3} \)

P5.8 Differentiate the following:

(1) \( y = ax^3 + 6 \) 
(2) \( y = 13x^{\frac{3}{2}} - c \) 
(3) \( y = 12x^{\frac{1}{2}} + c^{\frac{1}{2}} \)

(4) \( y = c^{\frac{3}{4}}x^{\frac{3}{2}} \) 
(5) \( u = \frac{ax^n + 1}{c} \) 
(6) \( y = 1.18t^2 + 22.4 \)

P5.9 Differentiate the following expressions:

(a) \( u = 1 + x + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \cdots \)

(b) \( y = ax^2 + bx + c \) 
(c) \( y = (x + a)^3 \)

P5.10 Use the product rule to find the following derivatives:

(1) If \( w = t(a - \frac{1}{2}bt) \), find \( \frac{dw}{dt} \).

(2) Find the derivative of \( y = (x + \sqrt{-1})(x - \sqrt{-1}) \).

(3) Differentiate \( y = (197x - 34x^2)(7 + 22x - 83x^3) \).

(4) If \( x = (y + 3)(y + 5) \), what is \( \frac{dx}{dy} \)?

(5) Differentiate \( y = 1.3709x(112.6 + 45.202x^2) \).

P5.11 Find the derivative of the following rational functions:

P5.12 Differentiate the following functions:

(1) \( y = \sqrt{x^2 + 1} \) 
(2) \( y = \sqrt{x^2 + a^2} \) 
(3) \( y = \frac{1}{\sqrt{x + a}} \)

(4) \( y = \frac{a}{\sqrt{a - x^2}} \) 
(5) \( y = \frac{\sqrt{x^2 - a^2}}{x^2} \) 
(6) \( y = \frac{\sqrt{x^2 + a}}{\sqrt{x^2 + a}} \)

(7) \( y = \frac{a^2 + x^2}{(a + x)^2} \) 
(8) \( y = \frac{1}{\sqrt{x^2}} \) 
(9) \( y = \frac{\sqrt{1 - x^2}}{1 - x} \)

P5.13 Use the chain rule to solve the following problems:

(1) Find \( \frac{du}{dx} \), if \( u = \frac{1}{2}x^3 \), \( v = 3(u + u^2) \), and \( w = \frac{1}{u} \).

(2) Find \( \frac{dv}{dx} \), if \( y = 3x^2 + \sqrt{2} \), \( z = \sqrt{y + 1} \), and \( v = \frac{1}{\sqrt{3 + 4z}} \).

(3) Find \( \frac{dv}{dz} \), if \( y = \frac{3}{\sqrt{2}} \), \( z = (1 + y)^2 \), and \( u = \frac{1}{\sqrt{3 + z}} \).

P5.14 Differentiate the functions:

(1) \( y = \ln x^n \) 
(2) \( y = 3e^{-\frac{x}{x-1}} \)

(3) \( y = (3x^2 + 1)e^{-5x} \) 
(4) \( y = (3x^2 - 1)(\sqrt{x} + 1) \)

(5) \( y = \ln(x^n + a) \) 
(6) \( y = \frac{\ln(x + a)}{x + a} \) 
(7) \( y = ax^n \)

(8) \( y = \ln(axe^x) \) 
(9) \( y = (\ln ax)^3 \)

P5.15 Differentiate \( f(x) = b(e^{ax} - e^{-ax}) \).

P5.16 Find the derivative with respect to \( t \) of \( u(t) = at^2 + 2\ln t \).

P5.17 If \( y = n^t \), find \( \frac{d(\ln y)}{dt} \).

P5.18 Find the derivative of \( f(x) = \frac{1}{b}e^{kx} \).

P5.19 Find the derivative of \( y \) with respect to \( x \) for

(a) \( y = x^a \) 
(b) \( y = (e^x)^x \) 
(c) \( y = e^{x^x} \)

Hint: Recall that \( \ln(ab) = b\ln(a) \) and \( a = e^{\ln a} \), for \( a > 0 \).

P5.20 Differentiate the following functions with respect to \( \theta \).
(1) \( y = A \sin(\theta - \frac{\pi}{2}) \)  
(2) \( y = \sin^2 \theta \)  
(3) \( y = \sin 5\theta \)  
(4) \( y = \sin^3 \theta \)  
(5) \( y = 18 \cos(\theta + 6) \)  
(6) \( y = \ln \cos \theta \)

**P5.21** Differentiate \( y = \frac{1}{2} \cos(2\pi nt) \).

**P5.22** Differentiate the following functions:

(i) \( y = \sin ax \)  
(ii) \( y = \sec x \)  
(iii) \( y = \cos^{-1}(x) \)  
(iv) \( y = \tan^{-1}(x) \)  
(v) \( y = \sec^{-1}(x) \)  
(vi) \( y = \tan(x)\sqrt{3} \sec x \)

**P5.23** Find the derivatives of the following functions:

(1) \( y = \sin \theta \sin(2\theta) \)  
(2) \( y = a \tan^n(\theta^n) \)  
(3) \( y = e^x \sin^2 x \)  
(4) \( y = \sin \left( (2\theta + 3)^2 \right) \)  
(5) \( y = \theta^3 + 3 \sin(\theta + 3) - 3\sin^3 \theta - 3^\theta \)

**P5.24** The length of an iron rod varies with temperature. Let \( \ell(t) \) denote the length of the iron rod (in metres) at temperature \( t[^\circ C] \). We measure the length of the rod at different temperatures and determine it’s length is described by the function \( \ell(t) = \ell_o(1 + 0.000012t)[m] \), where \( \ell_o \) is the length of the rod at 0°C. Find the change of length of the rod per degree Celsius.

**P5.25** The power \( P[W] \) consumed by an incandescent light bulb when connected given by the equation \( P = aV^b \), where \( a \) and \( b \) are constants, and \( V \) is the voltage drop across its terminals.

Find the rate of change of the power with respect to the voltage. Calculate the change in power per volt at the following operating voltages \( V_1 = 100[V] \), \( V_2 = 110[V] \), and \( V_3 = 120[V] \) in the case of a light bulb for which \( a = 0.008264 \) and \( b = 2 \).

**P5.26** The frequency \( f \) of vibration of a string of diameter \( D \), length \( L \) and mass density \( \sigma \), stretched with a tension \( T \) is given by the formula

\[
f = \frac{1}{DL} \sqrt{\frac{\varphi T}{\pi \sigma}}.
\]

Find the rate of change of the frequency \( f \) with respect to each of the variables: \( D \), \( L \), \( \sigma \), and \( T \).

**P5.27** Find the rate at which following geometrical quantities vary as a function of the radius:

(a) the circumference of a circle of radius \( r \)  
(b) the area of a circle of radius \( r \)  
(c) the area of a sphere of radius \( r \)  
(d) the volume of a sphere of radius \( r \)

Find the rate of change of the circumference of a sphere with respect to \( x \).

**P5.28** The temperature \( T \) of the filament of an incandescent electric lamp is connected to the current \( I \) passing through the lamp by the relation

\[
I = a + bT + cT^2.
\]

Find an expression giving the variation of the current with respect to a variation in temperature.

**P5.29** The following formulae have been proposed to express the relation between the electric resistance \( R[\Omega] \) of a wire at the temperature \( t[^\circ C] \). The resistance \( R_o \) corresponds resistance of the wire at 0[^\circ C], and \( a \), \( b \), \( c \) are constants.

\[
R_1(t) = R_o(1 + at + bt^2), \quad R_2(t) = R_o(1 + at + bt), \quad R_3(t) = R_o(1 + at + bt^2)^{-1}.
\]

Find the rate of variation of the resistance with regard to temperature as given by each of these formulae.

**P5.30** The voltage \( V \) of a certain type of standard cell varies with temperature \( t[^\circ C] \) according to the relation

\[
V(t) = 1.4340[1 - 0.000814(t - 15) + 0.000007(t - 15)^2]. \quad [V]
\]

Find the change of voltage per degree, at 15°C, 20°C and 25°C.

**P5.31** The voltage necessary to maintain an electric arc of length \( \ell \) with a current of intensity \( I \) was found by Mrs. Ayrton to be

\[
V = a + b\ell + \frac{c + k\ell}{I},
\]

where \( a \), \( b \), \( c \), \( k \) are constants. Find an expression for the variation of the voltage \( (a) \) with regard to the length of the arc \( \ell \); \( (b) \) with regard to the strength of the current \( I \).

**P5.32** Find \( \frac{dy}{dx} \) and \( \frac{d^2y}{dx^2} \) for the following functions:

(1) \( y = 17x + 12x^2 \)  
(2) \( y = ax^2 + bx + cx^4 \)  
(3) \( y = \frac{x^2 + 4}{x + a} \)

**P5.33** Calculate \( \frac{dy}{dx} \) and \( \frac{d^2y}{dx^2} \) for the functions in P5.9.

**P5.34** Calculate second and third derivatives of the functions in P5.10.

**P5.35** The distance travelled by a body falling freely in space is described by the equation \( d = 16t^2 \), where \( d \) is in feet, and \( t \) is in seconds. Draw a curve showing the relation between \( d \) and \( t \). Determine the velocity of the body at the following times: \( t = 2[s], t = 4.6[s], \) and \( t = 0.01[s] \).

**P5.36** If \( x = v_1t - \frac{1}{2}gt^2 \), find \( \dot{x} \) and \( \ddot{x} \).
P5.37 If a body moves according to the law \( x(t) = 12 - 4.5t + 6.2t^2 \) [m], find its velocity and its acceleration when \( t = 4 \) [s].

P5.38 The angle of rotation \( \theta \) [rad] of a revolving wheel as a function of time \( t \) [s] is described by the equation \( \theta(t) = 2.1 - 3.2t + 4.8t^2 \). Find the angular velocity of the wheel when \( t = 1 \frac{1}{2} \) [s]. Find also its angular acceleration.

P5.39 A slider moves so that its position \( x \) in inches from its starting point is given by the expression \( x(t) = 6.8t^3 - 10.8t \). Find the expression for its velocity and its acceleration at all times. Find its velocity and its acceleration at \( t = 3 \) [s].

P5.40 The height of a rising balloon (in [km]) is given at any instant by the expression \( h(t) = 0.5 + \frac{1}{10} \sqrt[3]{t - 125} \), where \( t \) is in seconds. Find the velocity and the acceleration at any time.

P5.41 A stone is thrown downward into water and its depth \( p \) (from the French profondeur) in metres after \( t \) seconds is given by the expression \( p(t) = \frac{4}{1 + \sqrt{t}} + 0.8t - 1 \). Find its velocity and its acceleration functions. Find the velocity and the acceleration at \( t = 10 \) [s].

P5.42 A body’s position as a function of time \( t \) is described by \( x(t) = t^n \), where \( n \) is a constant. Find the value of \( n \) when the velocity is doubled from the 5th to the 10th second. Find it also when the velocity is numerically equal to the acceleration at the end of the 10th second.

P5.43 Draw the graph of the function \( f(x) = \frac{2}{5}x^2 - 5 \) by hand using a scale of millimetres. Measure the slope of the function approximately from the graph at three different values of \( x \).

Next, find the derivative of the function and evaluate it at the same values of \( x \). See whether the slopes obtained from the derivative agree with the slopes measured graphically.

P5.44 Find the slope of the function \( f(x) = 0.12x^3 - 2 \) at \( x = 2 \).

P5.45 Given the function \( f(x) = (x-a)(x-b) \), find the \( x \) where \( f'(x) = 0 \).

P5.46 Find \( \frac{dy}{dx} \) for the function \( y = x^3 + 3x \) and calculate the numerical values of \( \frac{dy}{dx} \) for the points \( x = 0 \), \( x = \frac{1}{2} \), \( x = 1 \), and \( x = 2 \).

P5.47 Draw the graph of the function \( f(x) = be^{-\frac{x}{t}} \), with \( b = 12 \) and \( T = 8 \). Evaluate the function at various values of \( t \) from 0 to 20.

P5.48 The following equations give very similar graphs:

1. \( y = \frac{ax}{x+b} \)
2. \( y = a(1 - e^{-\frac{x}{T}}) \)
3. \( y = \frac{2a}{\pi} \tan^{-1}\left(\frac{x}{b}\right) \)

Obtain the graphs of these functions, taking \( a = 100 \) [mm] and \( b = 30 \) [mm].

P5.49 Differentiate the three functions from P5.48.

P5.50 Plot the curve \( y(\theta) = 100\sin(\theta - 15^\circ) \) and show that the slope of the curve at \( \theta = 75^\circ \) is half the maximum slope.

P5.51 Consider the curve described by the equation \( x^2 + y^2 = 4 \). Find the values of \( x \) where the slope of the curve is 1. What are the \((x, y)\) coordinates of the points on the curve where the slope is 17?

P5.52 Find the slope of the curve with equation \( \frac{x}{3} + \frac{y^2}{5} = 1 \) and give the numerical value of the slope at \( x = 0 \) and \( x = 1 \).

P5.53 Differentiate the expression \( z = \frac{2}{3} - 2x^2y - 2y^2x + \frac{2}{7} \) with respect to \( x \) then with respect to \( y \).

P5.54 Find the derivatives with respect to \( x \), \( y \), and \( z \), of the expression \( x^2y - x^2y^2 + xyz^2 + x^2y^2z^2 \).

P5.55 Let \( r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2 \). Find the value of \( \frac{dx}{dr} + \frac{dy}{dr} + \frac{dz}{dr} \). Also find \( \frac{dx}{dr}^2 + \frac{dy}{dr}^2 + \frac{dz}{dr}^2 \).

P5.56 Find the total differential of \( y = u^v \).

P5.57 Find the total differential of the following expressions:

1. \( y = u^3 \sin v \)
2. \( y = (\sin x)^u \)
3. \( y = \ln u \)

P5.58 The equation of a tangent to the curve \( y = 5 - 2x + 0.5x^3 \) is of the form \( y = mx + b \), where \( m \) and \( b \) are constants. Find the value of \( m \) and \( b \) for the tangent to the curve at \( x = 2 \).

P5.59 Consider the line \( \ell_1 \) defined by the equation \( y = x \) and the line \( \ell_2 \) defined by \( y = 3x \). Use the equation \( \theta = \tan^{-1}\left(\frac{y}{x}\right) \) to find angle each line makes with x-axis. Find the angle of intersection between the lines.

P5.60 At what angle do the two curves

\[
y = 3.5x^2 + 2 \quad \text{and} \quad y = x^2 - 5x + 9.5
\]

intersect each other?

P5.61 Two tangent lines to the curve \( y = \pm \sqrt{25 - x^2} \) are drawn at points \( x = 3 \) and \( x = 4 \). Find the coordinates of the point where the tangent lines intersect and their angle of intersection.

P5.62 A straight line \( y = 2x - b \) touches the curve \( y = 3x^2 + 2 \) at one point. What are the coordinates of the point of contact and what is the value of \( b \)?

P5.63 Find the value(s) of \( x \) that make \( y \) maximum or minimum:
(1) \( y = \frac{x^2}{x+1} \)  \hspace{1cm} (2) \( y = \frac{x}{a^2 + x^2} \)  \hspace{1cm} (3) \( y = x^5 - 5x \)

P5.64 Find the maxima and minima of \( f(x) = x^3 + x^2 - 10x + 8 \).

P5.65 Given \( y = \frac{3}{5}x - cx^2 \), where find expressions for \( \frac{dy}{dx} \) and \( \frac{d^2y}{dx^2} \). Find the value of \( c \) which makes \( y \) a maximum or a minimum. Assume \( c > 0 \).

P5.66 Find how many maxima and minima there are in the curves:

(a) \( y = 1 - \frac{x^2}{2} + \frac{x^4}{24} \)  \hspace{1cm} (b) \( y = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^5}{240} \)

P5.67 Find the maxima and minima of

(1) \( y = 2x + 1 + \frac{5}{x^2} \)  \hspace{1cm} (2) \( y = \frac{3}{x^2 + x + 1} \)  \hspace{1cm} (3) \( y = \frac{5x}{2x^2 + x} \)

P5.68 Divide a number \( N \) into two parts in such a way that three times the square of one part plus twice the square of the other part will be a minimum.

P5.69 The efficiency \( u \) of an electric generator at different values of output power \( x \)[W] is expressed by the general equation:

\[ u = \frac{x}{a + bx + cx^2}, \]

where \( a \) is a constant depending on the energy losses in the iron core and \( c \) is a constant depending on the resistance of the copper wires. Find the value of the output power \( x \) at which the efficiency is maximum.

P5.70 Suppose the consumption of coal by a certain steamer is represented by the formula \( y = 0.3 + 0.001v^3 \), where \( y \) is the number of tons of coal burned per hour and \( v \) is the speed expressed in nautical miles per hour. The cost of wages, financing, and depreciation of that ship are together equal, per hour, to the cost of 1 ton of coal. What speed will make the total cost of a voyage of 1000 nautical miles a minimum? And, if coal costs $100 per ton, what will that minimum cost of the voyage amount to?

P5.71 Find the maxima and minima of

(1) \( f(x) = \pm \frac{5}{6} \sqrt{x(10-x)} \)  \hspace{1cm} (2) \( g(x) = 4x^3 - x^2 - 2x + 1 \)

P5.72 Find the minimum or maximum of

(1) \( y = x^2 \)  \hspace{1cm} (2) \( y = x^{\frac{1}{2}} \)  \hspace{1cm} (3) \( y = x^a^{\frac{1}{3}} \)

P5.73 Find the value of \( \theta \in [0, \pi] \) for which \( \sin \theta \cos \theta \) is a maximum.

P5.74 Find the local maximum and minimum of \( y = \theta \cos \theta \) that are closest to the origin.

P5.75 One day you get tired of eating vegetables from the store and you decide to register for a spot in a local community garden to grow some vegetables. When you sign up, you’re given \( p \) metres of fencing to enclose a rectangle of land for your use. Show that the area of the rectangle will be a maximum if each of its sides is equal to \( \frac{1}{2}p \).

P5.76 A 30[in]-long piece of string is joined in a loop and is stretched by 3 pegs so as to form a triangle. What is the largest triangular area that can be enclosed by the string?

P5.77 Plot the curve corresponding to the equation \( y = \frac{10}{x} + \frac{10}{x^2} \). Find \( \frac{dy}{dx} \) and deduce the value of \( x \) that will make \( y \) minimum. What is the value of \( y \) at the minimum?

P5.78 What is the smallest square that fits inside another square, such that all the corners of the inner square touch the sides of the outer square?

P5.79 Suppose have a sphere of radius \( R \) and you want to draw a cylinder with radius \( r \) that fits inside the sphere. What is the cylinder whose (a) volume is maximum, (b) whose lateral area is maximum, and (c) whose total area is a maximum.

P5.80 A spherical balloon is increasing in volume. When its radius is \( r \) feet, its volume is increasing at the rate of 4 cubic feet per second. At what rate is its surface area increasing?

P5.81 It was shown by Lord Kelvin that the speed of signalling through a submarine cable depends on the value of the ratio of the external diameter of the core to the diameter of the enclosed copper wire. If this ratio is called \( y \), then the number of signals \( s \) that can be sent per minute can be expressed by the formula

\[ s(y) = ay^2 \ln \frac{1}{y}, \]

where \( a \) is a constant depending on the length and the quality of the materials. Find the value of \( y \) that makes \( s \) maximum.

Congratulations, you just cleared the second stage of calculus problems!

**Integrals problems**

We’re now starting the third batch of calculus problems. Get ready for the integrals!

I must give you heads up to caffeinate properly and be well rested when you start working on these problems. Unlike the “recipe” approach we use to solve derivative problems, integral problems require a lot more thinking and a trial-and-error approach. Instead of following a predefined procedure like the optimization algorithm, you’ll have to mentally browse through the integration techniques that you know and see which one applies to the problem you’re solving.
I’m not going to lie to you and tell you calculating integrals is easy. It’s hard work. The good news is that you’ll get a little mental buzz every time you solve a problem.

P5.82 Explain the equation \( \frac{df}{dx} \int_a^x f(s) \, ds = f(x) \) in your own words.

P5.83 Calculate the following integrals:

1. \( \int \sqrt{4ax} \, dx \)
2. \( \int \frac{3}{x^4} \, dx \)
3. \( \int \frac{1}{a} x^3 \, dx \)
4. \( \int (x^2 + a) \, dx \)
5. \( \int 5x^2 \, dx \)
6. \( \int \frac{(x^2 + a)}{x + a} \, dx \)
7. \( \int (x + 3)^3 \, dx \)
8. \( \int (x + 2)(x - a) \, dx \)
9. \( \int (\sqrt{x} + \sqrt[3]{x})^3 \, dx \)
10. \( \int (\sin \theta - \frac{1}{2}) \frac{d\theta}{3} \)
11. \( \int \cos^2(a\theta) \, d\theta \)
12. \( \int \sin^3 \theta \, d\theta \)
13. \( \int \sin^2 a\theta \, d\theta \)
14. \( \int e^{3x} \, dx \)
15. \( \int \frac{1}{x + 1} \, dx \)

P5.84 Calculate the integrals of these two polynomials:

(a) \( \int (4x^3 + 3x^2 + 2x + 1) \, dx \)
(b) \( \int \left( \frac{ax}{2} + \frac{bx^2}{3} + \frac{cx^3}{4} \right) \, dx \)

P5.85 Calculate the integrals:

1. \( \int \sqrt{a^2 - x^2} \, dx \)
2. \( \int x \ln x \, dx \)
3. \( \int x^3 \ln x \, dx \)
4. \( \int e^x \cos x \, dx \)
5. \( \int \frac{1}{x} \cos(\ln x) \, dx \)
6. \( \int x^2 e^x \, dx \)
7. \( \int \left( \frac{\ln x}{x} \right)^a \, dx \)
8. \( \int \frac{1}{x} \ln x \, dx \)
9. \( \int \frac{1}{x\sqrt{a - bx^2}} \, dx \)

P5.86 Split into partial fractions:

1. \( \frac{3x + 5}{(x - 3)(x + 4)} \)
2. \( \frac{3x - 4}{(x - 1)(x - 2)} \)
3. \( \frac{3x + 5}{x^2 + x - 12} \)
4. \( \frac{x + 1}{x^2 - 7x + 12} \)
5. \( \frac{x - 8}{(2x + 3)(3x - 2)} \)
6. \( \frac{x^2 - 13x + 26}{(x - 2)(x - 3)(x - 4)} \)
7. \( \frac{x^2 - 3x + 1}{(x - 1)(x + 2)(x - 3)} \)
8. \( \frac{5x^2 + 7x + 1}{(2x + 1)(3x - 2)(3x + 1)} \)

P5.87 Calculate these integrals:

1. \( \int \frac{5x + 1}{x^2 + x - 2} \, dx \)
2. \( \int \frac{(x^2 - 3)}{x^3 - 7x + 6} \, dx \)
3. \( \int \frac{b}{x^2 - a^2} \, dx \)
4. \( \int \frac{4x}{x^4 - 1} \, dx \)
5. \( \int \frac{x}{1 - x^4} \, dx \)

P5.88 Find the area under \( f(x) = x^2 + x - 5 \) between \( x = 0 \) and \( x = 6 \).

P5.89 Find the area under the parabola \( y = 2a\sqrt{x} \) from \( x = 0 \) to \( x = a \).

P5.90 Find the area under the sine curve from \( x = 0 \) to \( x = \pi \).

P5.91 Find the area under the curve \( y = \sin^2 x \) from \( x = 0 \) to \( x = \pi \).

P5.92 Find the area between the curves \( f(x) = x^2 + x^2 \) and \( g(x) = x^2 - x^2 \) from \( x = 0 \) to \( x = 1 \).

P5.93 What is the area under the graph of \( f(x) = x^3 - \ln x \) between \( x = 0 \) and \( x = 1 \)?

P5.94 Find the area of the portion of the curve \( xy = a \) included between \( x = 1 \) and \( x = a \).

P5.95 A certain curve has the equation \( y = 3.42e^{0.21x} \). Find the area included between the curve and the \( x \)-axis, from \( x = 2 \) to \( x = 8 \).

P5.96 Find the general solution to the following differential equations:

(a) \( \frac{dy}{dx} = \frac{1}{4} x \)
(b) \( \frac{dy}{dx} = \cos x \)
(c) \( \frac{dy}{dx} = 2x + 3 \)

Choose the additive constant of the general solution to obtain a specific solution \( y(x) \) that satisfies \( y(0) = 1 \).

Hint: The solution to a differential equation is a function \( y : \mathbb{R} \to \mathbb{R} \).

P5.97 Solve the differential equation \( f''(x) + 2f'(x) + f(x) = 0 \) for the unknown function \( f : \mathbb{R} \to \mathbb{R} \) and choose the coefficients in the general solution to obtain a specific solution that satisfies the initial conditions \( f(0) = 1 \) and \( f'(0) = 1 \).

Hint: The solutions have two independent parts of the form \( e^{-\lambda x} \) and \( xe^{-\lambda x} \).

P5.98 Calculate the length of the curve \( f(x) = \frac{1}{2} x^2 \) between \( x = 0 \) and \( x = 1 \).

Hint: This problem requires a long integral calculation. Start by rewriting \( \sqrt{1 + x^2} = \frac{1 + x^2}{\sqrt{1 + x^2}} \). Lookup the derivative formula for \( \sinh^{-1}(x) \). Use integration by parts and the self-referential trick from page ??.
P5.99 The voltage coming out of a North American electric wall outlet is described by the equation $V(t) = 155.57 \cos(\omega t)[V]$. The average squared voltage is calculated using the integral $V_{av}^2 = \frac{1}{T} \int_0^T V(t)^2 \, dt$. Calculate the root-mean-squared voltage $V_{rms} \equiv \sqrt{V_{av}^2}$. Hint: Make the substitution $\tau = \omega t$ and recall $\frac{1}{T} = \frac{T}{\omega \pi}$.

P5.100 Find the volume generated by the curve $y = \sqrt{1 + x^2}$ between $x = 0$ and $x = 4$ as it revolves about the $x$-axis.

P5.101 Find the volume generated by $\sin(x)$ revolving about the $x$-axis from $x = 0$ to $x = \pi$. Find also the surface area of this solid of revolution.

P5.102 Find the volume generated by the curve $y = \frac{x}{2} \sqrt{x(10-x)}$ between $x = 0$ and $x = 10$ as it rotates about the $x$-axis.

Stage 4 complete. Good job getting this far! I'm impressed with you!

Sequences and series problems

We're now entering the home stretch of the calculus problem set. You might want to revisit the definitions of convergence for sequences and series and review the various convergence tests.

P5.103 Determine whether the following sequences converge or diverge. If a sequence converges, state its limit.

(a) $a_n = \frac{2n + 3}{n + 1}$
(b) $b_n = \frac{n + 20}{\sqrt{n^2 - 10n}}$
(c) $c_n = \sqrt[n]{n} \equiv n^{\frac{1}{n}}$
(d) $d_n = (-1)^n$

Hint: For (c), recall $e^{\ln(y)} = y$ for all $y > 0$.

P5.104 Prove that the series $\sum_n \frac{1}{n^2}$ converges.

Hint: Use the direct comparison test with $b_n = \frac{1}{2\pi}$.

P5.105 Prove the series $\sum_n \frac{n^2}{2n^2 + 1}$ diverges using the divergence test.

P5.106 State whether the following series converge or diverge:

(1) $\sum_{n=1}^{\infty} \frac{\ln n}{n}$
(2) $\sum_{n=1}^{\infty} \frac{0.7}{n^2}$
(3) $\sum_{n=1}^{\infty} \frac{3}{3^{1/n} + 3}$
(4) $\sum_{n=1}^{\infty} \frac{1}{n^{1.1}}$
(5) $\sum_{n=1}^{\infty} \frac{1}{1 + n^2}$
(6) $\sum_{n=1}^{\infty} \frac{2n}{n^2 + 2}$

P5.111 State whether the series converge absolutely, converge conditionally, or diverge.

(1) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n^2 + 2}$
(2) $\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n}$
(3) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln n}$

P5.112 Use the $n^{th}$ root test or the ratio test to see whether the following series converge:

(1) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$
(2) $\sum_{n=1}^{\infty} \frac{n^2}{n^n}$
(3) $\sum_{n=1}^{\infty} \frac{(n!)^2}{n^n}$

P5.113 Figure out which test you need to use and determine if the following series converge or diverge:

(1) $\sum_{n=0}^{\infty} \frac{n}{n^2 + 4}$
(2) $\sum_{n=0}^{\infty} \frac{n}{(n^2 + 4)^2}$
(3) $\sum_{n=0}^{\infty} \frac{n!}{8n}$
(4) $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2 + 4}}$
(5) $\sum_{n=0}^{\infty} \frac{\sin^3(n)}{n^2}$
(6) $\sum_{n=0}^{\infty} \frac{n}{e^n}$

P5.107 State whether the following series converge or diverge:

(1) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2 + n}$
(2) $\sum_{n=1}^{\infty} \frac{(-1)^n 2n}{3n - 2n}$
(3) $\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n}$

Hint: Use the alternating series test.

P5.108 Check whether the following series converge or diverge:

(1) $\sum_{n=1}^{\infty} \frac{2^n}{3^n + 1}$
(2) $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$
(3) $\sum_{n=1}^{\infty} \frac{2^n}{3^n + 4^n}$

Hint: Use the alternating series test.

P5.109 Find the sum of $\frac{2}{3} + \frac{1}{3} + \frac{1}{12} + \frac{1}{24} + \cdots$.

P5.110 Calculate the values of the following infinite series:

(1) $\sum_{n=0}^{\infty} \frac{2^n}{3^n + 1}$
(2) $\sum_{n=0}^{\infty} \left( \frac{2}{3^n} + \frac{4}{5^n} \right)$
(3) $\sum_{n=1}^{\infty} \left( \frac{2}{3} \right)^n$
CALCULUS

\[ \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} \quad \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{(2n)!} \quad \sum_{n=0}^{\infty} \frac{6^n}{n!} \]

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}} \quad \sum_{n=1}^{\infty} \frac{2^n 3^{n-1}}{n!} \quad \sum_{n=1}^{\infty} \sin(1/n) \]

P5.114 Calculate the Maclaurin series of the function \( f(x) = \frac{1}{1+x} \).

P5.115 Find the Maclaurin series for the following functions:

(a) \( f(x) = \frac{1}{1-x^2} \)  
(b) \( g(x) = e^{-x} \)  
(c) \( h(x) = x^2 \cos(x^2) \)

Hint: For (b) and (c), you don’t need to compute all the derivatives: use algebraic manipulations starting from a Maclaurin series that you know.

P5.116 Find the Taylor series expansions at the point \( x = a \) specified.

(a) \( f(x) = e^x \), around \( a = 5 \).  
(b) \( g(x) = \sin(x) \), at \( a = 10 \).

P5.117 Find the radius of convergence for the following power series:

\( \sum_{n=0}^{\infty} nx^n \)  
\( \sum_{n=0}^{\infty} \frac{x^n}{n!} \)  
\( \sum_{n=0}^{\infty} \frac{x^n}{n(n+1)} \)  
\( \sum_{n=0}^{\infty} \frac{(1)^n}{n!} x^n \)  
\( \sum_{n=0}^{\infty} \frac{x^n}{1 + 3^n} \)  
\( \sum_{n=0}^{\infty} \frac{(-1)^n}{n^3 3^n} x^{2n} \)

P5.118 Find a series for each function, using the formula for Maclaurin series and algebraic manipulation as appropriate.

\( 2^n \)  
\( \ln(1 + x) \)  
\( \ln \left( \frac{1 + x}{1 - x} \right) \)  
\( \sqrt{1 + x} \)  
\( \frac{1}{1 + x^2} \)  
\( \tan^{-1}(x) \)

Links

Below are some links to more calculus problems with solutions.

[ Lot’s of solved calculus examples by Larry Perez ]
http://www.saddleback.edu/faculty/lperez/algebra2go/calculus/calc3A.pdf

[ Try the problems in the calculus textbook by Gilbert Strang ]

Appendix A

Answers and solutions

Chapter 1 solutions

Answers to problems

P1.1 \( x = \pm 4 \).  
P1.2 \( x = A \cos(\omega t + \phi) \).  
P1.3 \( x = \frac{ab}{a+b} \).  
P1.4 (a) 2.7975.  
(b) 1024.  
(c) –8.373.  
(d1) 11.  
P1.5 (a) \( x = \sqrt{2} \).  
(b) \( x = \left( \frac{3}{2} + 2\pi n \right) \) for \( n \in \mathbb{Z} \).  
P1.6 (1) c.  
(2) l.  
(3) \( \frac{9\pi}{2} \).  
(4) a.  
(5) \( \frac{b}{a} \).  
(6) \( x^2 - ab \).  
P1.7 (1) \( x = \frac{3}{4} \) \( (x+2) \).  
(2) \( 3x(x-3)(x+3) \).  
(3) \( (x+3)(6x-7) \).  
P1.8 $0.05 \).  
P1.9 13 people, 30 animals.  
P1.10 5 years later.  
P1.11 girl = 80 nuts, boy = 40 nuts.  
P1.12 Alice is 15.  
P1.13 18 days.  
P1.14 After 2[h].  
P1.16 \( \varphi = \frac{1 + \sqrt{5}}{2} \).

P1.17 \( x = \frac{3-\sqrt{41}}{2} \).  
P1.18 No real solutions if \( 0 < m < 8 \).  
P1.19 (1) \( e^x \).  
(2) \( \frac{3^{x^2}}{x^2} \).  
(3) \( \frac{1}{x^2} \).  
(4) \( \frac{1}{x^2} \).  
(5) \( \frac{3}{4} \).  
(6) \( \ln(x+1) \).  
P1.20 (1) \( \frac{1}{x} \).  
(2) \( \frac{1}{1+x} \).  
(3) \( \frac{3}{x+1} \).  
P1.21 (1) \( x \in (4,\infty) \).  
(2) \( x \in [3,6] \).  
(3) \( x \in (-\infty,-1) \cup \left( \frac{1}{2},\infty \right) \).  
P1.22 For \( n > 250 \), Algorithm Q is faster.  
P1.23 10 cm.  
P1.24 22.51 in.  
P1.25 \( h = \sqrt{3.33^2 - 1.44^2} = 3 \text{ m}. \)  
P1.26 The opposite side in has length 1.  
P1.27 \( x = \sqrt{3}, \) \( y = 1, \) and \( z = 2 \).  
P1.28 \( x = \frac{\tan 25^\circ \tan 25^\circ}{\tan 25^\circ - \tan 25^\circ} \).  
P1.29 \( d = \frac{\tan 25^\circ - \tan 25^\circ}{\tan 25^\circ} \).  
P1.30 \( t = \theta \cos^2 \theta \).  
P1.31 \( \sin \theta \cos^2 \theta = \frac{1}{8} \cos 3\theta \).  
P1.32 \( a = \sqrt{3} \).  
P1.33 \( P_c = 16 \tan(22.5^\circ) \).  
P1.34 (a) \( h = a \sin \theta \).  
(b) \( A = \pi a \sin \theta \).  
(c) \( c = \sqrt{a^2 + b^2 - 2ab \cos(180^\circ - \theta)} \).  
P1.35 \( c = \frac{a \sin 30^\circ}{2} \approx 14.7 \).  
P1.36 \( B = 44.8^\circ, C = 110.2^\circ \).  
P1.37 \( v = 742.92 \text{ km/h} \).  
P1.38 1.33 cm.  
P1.39 \( a = 9.55 \).  
P1.40 \( \frac{2}{3} (\pi^2 - 2\pi^2) = 18.85 \text{ cm}^2 \).  
P1.41 \( \ell_{opt} = 7.83 \text{ m} \).  
P1.42 \( \theta = 120^\circ \).  
P1.43 \( a = \frac{1}{4} \sin 15^\circ \).  
P1.44 7 cm.  
P1.45 A_{sect} = 5c = 10. \)  
P1.46 \( V_{box} = 1.639 \text{ L}. \)  
P1.47 \( V = 300,000 \text{ L}. \)  
P1.48 315,000 L.  
P1.49 4000 L.  
P1.50 \( d = \frac{1}{2} (35 - 2\sqrt{21})\).  
P1.51 Use a rope of length $\sqrt{2}$.  
P1.52 20L of water.  
P1.53 \( h = 7.84375 \text{ in}. \)  
P1.54 \( 1 + 2 + \cdots + 100 = 50 \times 101 = 5050 \).  
P1.55 \( x = -2 \) and \( y = 2 \).  
P1.56 \( x = 1, \) \( y = 2, \) and \( z = 3 \).  
P1.57 $\pi 12 \).  
P1.58 20%.  
P1.59 $\pi 6501.93 \).  
P1.60 0.14 s.  
P1.62 \( r = 34.625 \text{ min} \).  
P1.63 \( V(0.01) = 15.58 \text{ volts}. \)  
\( V(0.1) = 1.642 \text{ volts}. \)
Solutions to selected problems

P1.9 Let \( p \) denote the number of people and \( a \) denote the number of animals. We are told \( p + a = 43 \) and \( a = p + 17 \). Substituting the second equation into the first, we find \( p + (p + 17) = 43 \), which is equivalent to \( 2p = 26 \) or \( p = 13 \). There are 13 figures of people and 30 figures of animals.

P1.10 We must solve for \( x \) in \( 35 + x = 4(3 + x) \). We obtain \( 35 + x = 20 + 4x \), then \( 15 = 3x \), so \( x = 5 \).

P1.12 Let \( A \) be Alice’s age and \( B \) be Bob’s age. We’re told \( A = B + 5 \) and \( A + B = 25 \). Substituting the first equation into the second we find \((B + 5) + B = 25\), which is the same as \( 2B = 20 \), so Bob is 10 years old. Alice is 15 years old.

P1.13 The first shop can bind 4500/30 = 150 books per day. The second shop can bind 4500/150 = 100 books per day. The combined production capacity rate is 150 + 100 = 250 books per day. It will take 4500/250 = 18 days to bind the books when the two shops work in parallel.

P1.14 Let \( x \) denote the distance the slower plane will travel before the two planes meet. Let \( t_{\text{meet}} \) denote the time when they meet, as measured from moment when the second plane departs. The slower plane must travel a distance \( x \) [km] in time \( t_{\text{meet}} \) [h], so we have \( t_{\text{meet}} = \frac{x}{600} \). The faster plane is 600[km] behind when it departs. It must travel a distance \((x + 600)\) [km] in the same time so \( t_{\text{meet}} = \frac{x + 600}{600} \). Combining the two equations we find \( \frac{x}{600} = \frac{x + 600}{600} \). After cross-multiplying we find \( 900x = 600x + 600^2 \), which has solution \( x = 1200\) [km]. The time when the planes meet is \( t_{\text{meet}} = 2\) [h] after the departure of the second plane.

P1.15 This is a funny nonsensical problem that showed up on a school exam.

P1.18 Using the quadratic formula, we find \( x = \frac{m \pm \sqrt{m^2 - 8m}}{2} \). If \( m^2 - 8m \geq 0 \) the solutions are real. If \( m^2 - 8m < 0 \), the solutions will be complex numbers. Factoring the expressions and plugging in some numbers, we observe that the \( m^2 - 8m = m(m - 8) < 0 \) for all \( m \in (0, 8) \).

P1.20 For \((3)\), \( 4 \frac{1}{2} + 1 \frac{1}{2} = 7 + \frac{63}{32} = \frac{56}{32} + \frac{63}{32} = \frac{119}{32} = 3 \frac{23}{32} \).

P1.21 For \((3)\), \( 1 \frac{1}{4} + 3 \frac{1}{4} = 7 + \frac{63}{32} = \frac{56}{32} + \frac{63}{32} = \frac{119}{32} = 3 \frac{23}{32} \).

P1.22 The running time of Algorithm Q grows linearly with the size of the problem, whereas Algorithm P’s running time grows quadratically. To find the size of the problem when the algorithms take the same time we solve \( \Phi(n) = \Theta(n) \). The solution is \( n = 250 \). For \( n > 250 \) the linear-time algorithm (Algorithm Q) will take less time.

P1.26 Solve for \( b \) in Pythagoras’ formula \( c^2 = a^2 + b^2 \) with \( c = \varphi, \) and \( a = \sqrt{2} \). The triangle with sides 1, \( \sqrt{2} \), and \( \varphi \) is called Kepler’s triangle.

P1.27 Use Pythagoras’ theorem to find \( x \). Then use \( \cos(30^\circ) = \frac{\sqrt{3}}{2} \) to find \( z \). Finally use \( \sin(30^\circ) = \frac{1}{2} \) to find \( y \).

P1.28 Consider the right-angle triangle with base \( x \) and opposite side 2000. Looking at the diagram we see that \( \theta = 24^\circ \). We can then use the relation \( \tan 24^\circ = \frac{2000}{x} \) and solve for \( x \).

P1.29 Consider the two right-angle triangles drawn in the figure. From the triangle with angle 25° we know \( \tan 25^\circ = \frac{h}{800 + d} \). From the triangle with angle 20° we know \( \tan 20^\circ = \frac{h}{1800 - d} \). We isolate \( h \) in both equation we can eliminate \( h \) by equating: \((1800 + d) \tan 25^\circ = \tan 20^\circ (800 + d) \). Solving for \( d \) we find \( d = \frac{1800 \tan 20^\circ - 800 \tan 25^\circ}{\tan 25^\circ - \tan 20^\circ} \) = 2756.57 m. Finally we use \( \tan 25^\circ = \frac{h}{800 + d} \) again to obtain \( 2756.57 \) m.

P1.31 We know \( \sin^2(\theta) = \frac{1}{2} (1 - \cos(2\theta)) \) and \( \cos^2(\theta) = \frac{1}{2} (1 + \cos(2\theta)) \) so their product is \( \frac{1}{2} (1 - \cos(2\theta) \cos(2\theta)) \). Note \( \cos(2\theta) \cos(2\theta) = \cos^2(2\theta) \). Using the power-reduction formula on the term \( \cos^2(2\theta) \) leads to the final answer \( \sin^2 \theta \cos^2 \theta = \frac{1}{4} (1 - \cos(4\theta)) \).

P1.32 The internal angles of an equilateral triangle are all 60°. Draw three radial lines that connect the centre of the circle to each vertex of the triangle. The equilateral triangle is split into three obtuse triangles with angle measures 30°, 30°, and 120°. Splitting each of these obtuse sub-triangles down the middle to obtain six right-angle triangles with hypotenuse 1. The side of the equilateral triangle is equal to two times the base of the right-angle triangles \( a = 2 \cos(30^\circ) = \sqrt{3} \). To find the area, we use \( A_\Delta = \frac{1}{2} ab \), where \( h = 1 + \sin(30^\circ) \).

P1.33 Split the octagon into 8 isosceles triangles. The height of each triangle will be \( 900 \). After cross-multiplying we find \( 900 \) [cm] = 0. The vertical part of the rope has length \( 800 + d \) again.

P1.37 Initially the horizontal distance between you and the plane is \( d = \frac{2000}{80} \) [m]. After 10 seconds, the distance is \( d_2 = \frac{500}{80} \) [m]. Velocity change in distance divided by the time \( v = \frac{500 - 200}{10} \) = 30 m/s. To convert m/s into km/h, we must multiply by the appropriate conversion factors: 20636.36 m/s × \( \frac{1 \text{ km}}{1000} \) × \( \frac{1 \text{ h}}{3600} \) = 742.92 km/h.

P1.39 Using the law of cosines for the angles \( \alpha_1 \) and \( \alpha_2 \) we obtain the equations \( \sum = 8^2 + 12^2 - 2(8)(12) \cos \alpha_1 \) and \( 11^2 = 4^2 + 12^2 - 2(4)(12) \cos \alpha_2 \) from which we find \( \alpha_1 = 34.09^\circ \) and \( \alpha_2 = 66.03^\circ \). In the last step we use the law of cosines again to obtain \( x^2 = 8^2 + 4^2 - 2(8)(4)\cos(34.09^\circ + 66.03^\circ) \).

P1.41 The length of the horizontal part of the rope is \( L_\phi = 4 \sin 40 \). The circular portion of the rope that hugs the pulley has length \( \frac{1}{2} \) of the circumference of a circle with radius \( r = 50 \) cm = 0.5 m. Using the formula \( C = 2\pi r \), we find \( L_\phi = \frac{1}{2} \pi (0.5)^2 = \frac{\pi}{4} \). The vertical part of the rope has length \( L_\phi = 4 \cos 40 + 2 \). The total length of the rope is \( L_\phi + L_\phi = 7.83 \) m.

P1.42 We didn’t really cover these concepts in the book, but since we’re on the topic let’s define some vocabulary. The complement of an acute angle is its defect from a right angle, that is, the angle by which it falls short of a right angle. (1) Two angles are complementary when their sum is 90°. The supplement of an angle is its defect from two right angles, that is, the angle by which it falls short
of two right angles. (ii) Two angles are supplementary, when their sum is 180°. Angles which are complementary or supplementary to the same angle are equal to one another.

We'll now use these facts and the diagram below to find the angle θ.

![Diagram of angles](image)

The angle α is vertically opposite to the angle 60° so α = 60°. The angle β is supplementary to the angle 100° so β = 180° − 100° = 80°. The angles γ = β = 80° because the angles are vertically opposite. The sum of the angles in a triangle is 180° so γ = 180° − α − β = 40°. The two horizontal lines are parallel so the diagonally cutting line makes the same angle with them: γ' = γ = 40°. The angle θ is supplementary to the angle γ' so θ = 180° − 40° = 120°.

**P1.43** The base of this triangle has length 2r and each side has length R + r. If you split this triangle through the middle, each half is a right triangle with an angle at the centre 360°/2 = 15°, hypotenuse R + r, and opposite side r. We therefore have sin 15° = R/(R + r). After rearranging this equation, we find R = 1 − sin 15° = 2.8637.

**P1.45** The area of a rectangle is equal to its length times its height Arect = ℓh.

**P1.46** The box’s volume is V = w × h × ℓ = 10.5 × 7 × 22.3 = 1639 cm³ = 1.639 L.

**P1.48** The total capacity is 15 × 6 × 5 = 450 m³. If 30% of its capacity is spent, then 70% of the capacity remains: 315 m³. Knowing that 1 m³ = 1000 L, we find there are 315,000 L in the tank.

**P1.49** The first tank has 1/4 × 4000 = 1000 L in it. The second tank has three times more water, so 3000 L. The total is 4000 L.

**P1.50** Let’s define w and h to be the width and the height of the hole. Define also d to be the distance of the hole from the sides of the lid. The statement of the problem dictates the following three equations must be satisfied w + 2d = 40, h + 2d = 30, and wh = 500. After some manipulations we find w = 5(1 + √2), h = 5(√21 − 1), and d = 1/2(35 − 5√21).

**P1.51** The amount of wood in a pack of wood is proportional to the area of a circle A = πr². The circumference of this circle is equal to the length of the rope C = ℓ. Note the circumference is proportional to the radius C = 2πr. If we want double the area, we need the circle to have radius √2r, hence the circumference will need to be √2 times larger. Hence, if we want a pack with double the wood, we need to use a rope of length √2ℓ.

**P1.52** In 10 L of 60% solution of acid there are 6 L of acid and 4 L of water. A 20% solution of acid will contain four times as much water as it contains acid, so 6 L acid and 24 L water. Since the 10 L we are starting from already contains 4 L of water, we must add 20 L.

**P1.53** The document must have a 768/1004 aspect ratio, so its height must be 6 × 1004/768 = 7.84375 in.

**P1.54** If we rewrite 1 + 2 + 3 + ⋯ + 98 + 99 + 100 by pairing numbers, we obtain the sum (1 + 100) + (2 + 99) + (3 + 98) + ⋯. This list has 50 terms and each term has the value 101. Therefore 1 + 2 + 3 + ⋯ + 100 = 50 × 101 = 5050.

**P1.59** An nAPR of 12% means the monthly interest rate is 12%/12 = 1%. After 10 years you’ll owe $5000(1.01)^{120} = $16501.93. Yikes!

**P1.60** The graph of the functions are shown in Figure A.1. Observe that f(x) has decreased to 37% of its initial value when x = 2. The increasing exponential g(x) has reached 63% of its maximum value at x = 2.

![Graphs](image)

(a) The graph of f(x).

(b) The graph of g(x).

**Figure A.1:** The graphs of the two functions from P1.60.

**P1.61** We’re looking for the time t such that Q(t)/Q₀ = 1/2, which is the same as e⁻⁵t = 0.5. Taking logarithms of both sides we find −5t = ln(0.5) and solving for t we find t = 0.14 s.

**P1.62** We are told that T(24)/T₀ = 1/3 = e⁻²⁴/τ, which we can rewrite as ln(1/3) = −24/τ. Solving for τ, we find τ = 24/ln(1/3) = 34.625 min. To find the time to reach 1% of the initial temperature, we must solve for t in T(t)/T₀ = 0.01 = e⁻¹/34.625. We find t = 159.45 min.

**P1.64** “There exists at least one banker who is not a crook.” Another way of saying the same thing is “not all bankers are crooks”—just most of them.

**P1.65** Everyone in Monsanto’s leadership ought to burn in hell, forever.

**P1.66** (a) Investors with money but without connections. (b) Investors with connections but no money. (c) Investors with both money and connections.

## Chapter 2 solutions

### Answers to problems

**P2.1** The particle is gaining speed and its motion is a UAM. **P2.2** Not UVM.

**P2.3** (1) Speed is decreasing. (2) Speed is increasing. **P2.4** UAM before t = t₀ and UVM after t = t₀. **P2.5** v_A = 6[m/s], v_B = 6[m/s], v_C = 8[m/s],
P2.12 Your velocity during the first time interval (from $x$ for each step. Note the area is negative from D to E. (1) $v_i(0) = 0[m/s]$, $v_i(2) = 2[m/s]$, $v_i(3) = 4[m/s]$. (2) For $t > 2[s]$: $v(t) = 6 + 6(t-2) - 6t^2$ [m]. (3) When $x = 49[m]$, $t = 5[s]$. (4) $v = 12[m/s]$ when $x = 14[m]$. P2.11 (1) $v_{A-B} = 0[m/s]$, $v_{C-D} = 2[m/s]$ and $v_{E-F} = -5[m/s]$. (2) From $0[s]$ to $2[s]$ the squirrel is not running, from $2[s]$ to $6[s]$ it’s running forward, and from $6[s]$ to $9[s]$ it’s running backward. The squirrel changes direction at $t = 6[s]$. P2.12 (1) $v_i = 6[m/s]$. (2) $x(t) = 6t - t^2$ [m]. P2.13 (1) $x_1(t) = \frac{1}{2}t^2[m]$ and $x_2(t) = \frac{1}{2}t^2 + 3t + 4[m]$. (2) $x = 24[m]$ when $t = 4[s]$. P2.14 (1) $x_1 = 7[m]$ and $v_1 = 5[m/s]$. (2) $v(t) = 4t + 5$ $[m/s]$ and $a(t) = 4[m/s^2]$. (3) $x(5) = 82[m]$ and $v(5) = 25[m/s]$. P2.15 $v_i = 4\sqrt{2}[m/s]$ and $\Delta t = \sqrt{2}[s]$.

Solutions to problems

P2.1 The relation between velocity and time is linear and so $v$ increases with time. The rate at which the velocity increases is the acceleration which is constant.

P2.2 Your velocity during the first time interval (from $x = 0$ to $t = 2[s]$) is 1.5$[m/s]$. During the second time interval your velocity is 2.5$[m/s]$, and during the third it is 3$[m/s]$. Your run is not a UVM.

P2.3 When the velocity and the acceleration are in opposite directions, the car’s speed will decrease. When the velocity and the acceleration are in the same direction, the car’s speed will increase.

P2.4 Use $F = ma$. When $F$ is constant, $a$ is constant and the motion is UAM. When $F = 0$, $a = 0$ and the motion is UVM.

P2.5 To find the velocity, calculate the area under the graph and add the areas for each step. Note the area is negative from D to E.

P2.6 Differentiate the function $v(t)$ to find the velocity $v(t)$. Differentiate the function $a(t)$ to find the acceleration $a(t)$.

P2.7 Use $v_i^2 - v_f^2 = 2a(y_f - y_i)$ to find the acceleration. Use the general equation $y(t) = y_i + v_i t + \frac{1}{2}at^2$ and plug in $y_i = 4[m]$, $v_i = 0[m/s]$, and $a = -24.5[m/s^2]$ to find the height as a function of time.

P2.8 Differentiate the position with respect to $t$ to find $v(t)$, then differentiate $v(t)$ to find $a(t)$. Use $F = ma$ to find the force.

P2.9 (1) Calculate $a(t)$ from Newton’s $2^{nd}$ law to obtain $a(t) = 2[m/s^2]$. Integrate with respect to time to obtain the velocity and then integrate again to find the position. (2) Plug $t = 4[s]$ into $v(t)$. (3) Use the fourth equation to find the speed after the car has travelled 9[m].

P2.10 (1) Use integration to find $v(t) = \frac{3}{2}t^2[m/s]$ and $x(t) = \frac{1}{2}t^3[m]$, then plug in $t = 2[s]$. (2) To describe UAM staring at $t = 2[s]$, use the general equation $x(t) = x_2 + v_2(t - 2) + \frac{1}{2}a(t - 2)^2$, where $x_2 = 4[m]$ and $v_2 = 6[m/s]$ are the initial conditions at $t = 2[s]$. (3) Solve for $t$ in $x(t) = 49[m]$. (4) Use $v_i^2 - v_f^2 = 2a(x - x_2)$ with $v_i = 6[m/s]$, $v_f = 12[m/s]$, and $x_2 = 4[m]$ to find the displacement during the UAM.

P2.11 (1) In each time interval, divide $\Delta x$ by $\Delta t$ to find the velocity. (2) Increasing $x$ means the squirrel is running forward. Decreasing $x$ means it’s running backward.

P2.12 Use the fourth equation of motion to find $v_i = 6[m/s]$, then construct the position function $x(t) = x_i + v_it + \frac{1}{2}at^2$ using $x_i = 0[m]$, $v_i = 6[m/s]$, and $a = -2[m/s^2]$.

P2.13 Find the position functions of both dogs, then equate them to get an equation with $t$ unknown. Solve $x_1(t) = x_2(t)$ for $t$ to find the time when the dogs meet.

P2.14 Recall that the coefficient in front of $t$ in the general form of the position function is the initial velocity $v_i$ and the constant term is $x_i$. Use differentiation to find the velocity and acceleration functions and then evaluate them at $t = 5[s]$.

P2.15 Use the fourth equation of motion twice. First between $x = 2[m]$ and $x = 4[m]$ to find the acceleration $a = -4[m/s^2]$, then between $x = 0[m]$ and $x = 4[m]$ to find $v_i$. Construct the position as a function of time and find the time needed to reach $x = 4[m]$.

Chapter 3 solutions

Answers to problems

P3.1 (a) $\vec{v}_1 = 5\vec{e}_x + 0\vec{e}_y$; (b) $\vec{v}_2 = \sqrt{2}\vec{e}_x + 3\vec{e}_y$. (c) $\vec{v}_3 = \sqrt{2}\vec{e}_x - 3\vec{e}_y$ or $\sqrt{2}\vec{e}_x + 3\vec{e}_y$. P3.2 (a) $\vec{v}_1 = (17,32,10)$. (b) $\vec{v}_2 = (0,-10)$. (c) $\vec{v}_3 = (-4.33,2.5)$. P3.3 (a) $\vec{v}_1 = 9.06\vec{e}_x + 4.23\vec{e}_y$. (b) $\vec{v}_2 = 7\vec{e}_x - 3\vec{e}_y$. (c) $\vec{v}_3 = 3\vec{e}_x - 2\vec{e}_y + 3\vec{e}_z$. P3.4 (a) 34. (b) (0,1). (c) (7.34,3.4). P3.6 (1) 6. (2) 0. (3) -3. (4) (-2,1,1). (5) (3,-3,0). (6) (7,-5,1). P3.7 (1,2,5) or (1/5, 2/5, 1).

P3.8 (12, -4, -12). P3.9 (a) 21. (b) $\frac{1}{3}(5+i)$. (c) $2+i$. P3.10 (a) $x = 2i$. (b) $x = -16$. (c) $x = -1 - i$ and $x = -1 + i$. (d) $x = i$. (e) $x = -i$. (f) $x = 3i$. (g) $x = -\sqrt{3}i$. P3.11 (a) $\sqrt{5}$. (b) $\frac{1}{2}(-3+i)$. (c) $-5-i$. P3.12 52 weeks.

Solutions to selected problems

P3.7 See bit.ly/1ca8So for calculations.

P3.8 Any multiple of the vector $\vec{u}_1 \times \vec{u}_2 = (-3,1,3)$ is perpendicular to both $\vec{u}_1$ and $\vec{u}_2$. We must find a multiplier $t \in \mathbb{R}$ such that $t(-3,1,3) = (1,1,0)$. Computing the dot product we find $-3t + t = 8$, so $t = -4$. The vector we’re looking for is $(12,-4,-12)$. See bit.ly/1ly9fYt for calculations.

P3.12 We want the final state of the project to be 100% real: $p_f = 100$. Given that we start from $p_i = 100i$, we must have $e^{-i\omega t} = e^{-i\frac{\pi}{2}}$, which is the same as $\alpha(t) = \frac{\pi}{2}$. We can rewrite this equation as $h(t) = 0.2t^2 = \frac{\pi}{2}$ and solving for $t$ we find $t = \sqrt{\frac{\pi}{0.2}} = 52$[weeks].

Chapter 4 solutions

Answers to exercises

E4.1 7.5$[m/s]$.

Solutions to exercises

E4.1 Assume the initial velocity is in the $x$-direction. The initial momentum of the incoming ball is $\vec{p}_{in} = m\vec{v} = 3 \times 201 = 601$. The initial momentum of the stationary ball is zero. The momentum after the collision is $p_{out} = p_{1in} + p_{2in} = 601 + 0 = 601$. The two balls stick together so the total mass of the system after the collision is $M = 3 + 5 = 8$. The final velocity is $v_{out} = p_{out}/M = \frac{1}{8}(601,0) = (7.5,0)[m/s]$. 
Answers to problems

P4.1 (1) With $y$ pointing up $v_y(t) = v_{y0}t - \frac{1}{2}gt^2$, with $y$ pointing down $v_y(t) = -v_{y0}t + \frac{1}{2}gt^2$, where $v_{y0} = ||\vec{v}_0|| \sin \theta$. (2) The cat gets splashed! P4.2 (1) $\vec{F}_a$ points right and is perpendicular to the left face of the block, (2) $\vec{F}_a$ points up and is perpendicular to the bottom face of the block, and (3) $\vec{F}_g$ points left and is perpendicular to the right face of the block. P4.3 The particles have the same momentum. The first particle has twice the kinetic energy of the second one. P4.4 (1) $\vec{v}_0 = \vec{v}$, (2) $\vec{v}_0 = 3\vec{v}$, and (3) $\vec{v}_0 = 2\vec{v}$. P4.5 In position 1, there is only $K$. In position 2 there is $K$, $U_x$, and $U_y$. P4.6 The ball will have the same speed on Earth and on the Moon. P4.7 The ball will have higher speed on Earth. P4.8 They will have the same speed when they hit the ground.

P4.9 $|\vec{v}_{bottom}| > |\vec{v}_{top}|$. P4.10 The torque on $M$ is zero. The rod’s angular velocity will remain $\omega$. P4.11 Since the three pendulums’ strings have the same length $l$, they have the same period $T_1 = T_2 = T_3$. P4.12 $T_1 = 1.42[s]$, $T_2 = 1.74[s]$, and $T_3 = 2.01[s]$. P4.13 The pendulum swings through its centre position with greater speed on Earth. P4.14 Energies present: $K$, $U_r$, and $U_g$. Momenta present: $\vec{p}$ and $L$. P4.15 The period will decrease. P4.16 $F_{push} = 4.71[N]$. P4.17 $E_{lost} = K_i - K_f = 19.7[J]$. P4.18 (1) $v_2 = -5.4[m/s]$. (2) Not elastic. P4.19 (1) See solution. (2) $a_1 = 1[m/s^2]$, $a_2 = 0[m/s^2]$, and $a_3 = -1[m/s^2]$. (3) $F = 40[N]$. P4.20 $v_{bob} = \sqrt{2}v[m/s]$. P4.21 (1) $T = 4.9[N\cdot m]$. (2) $W = 9.6[J]$. P4.22 $m = 1.46[kg]$. P4.23 $\mu_0 = 0.24$. P4.24 (1) $h = d$. (2) $v_i = 19.8[m/s]$ and $t_{hit} = 1.435[s]$. P4.25 (1) $\mu_0 \geq \frac{M}{M_1+m_2}$. (2) $\mu_0 \geq 0.421$. (3) $a = \frac{M \cdot L}{m_1 \cdot m_2 + m_7}$. P4.26 (1) $d = 4.41[m]$. (2) $\vec{v}_2 = 5.290[m/s]$. P4.27 (1) $F_{diss} = 158 \times 10^3[N]$. (2) The vertical acceleration is zero so the plane will maintain a horizontal trajectory. P4.28 $v = 10.2[m/s]$. P4.29 (1) $v_1 = (\sin(30) \cdot \cos(30)) \sqrt{2}g \cdot \mu_2$. (2) $v_1 = 2.24[m/s]$, $v_2 = 1.12[m/s]$ and $v_3 = 1.94[m/s]$. (3) $0.233[m]$. (4) The collision is elastic. P4.30 (1) $T = 3.6[N\cdot m]$. (2) $18.9 rev$. P4.31 $\mu = \frac{L}{r^2}$. P4.32 $\mu = \frac{L}{r^2}$. P4.33 Range is 0.65[m] greater on the summit than on the North Pole. P4.34 $f_{flight} = 2t_{flap} = 4.1[s]$. P4.35 $x(t) = 2t^2 + 10t + 21$ in metres. P4.36 Loses contact at $R = 2.44$. P4.37 Upward $F_{friction} >$ stationary $F_{f_r} >$ downward. P4.38 The coin further from the centre will fly off first. P4.39 The pulley with the larger radius $R$ will be spinning faster and have more $K$. P4.40 (1) $F = 3000[N]$ per wheel. (2) $T = 180[N\cdot m]$. (3) 2.21 turns. (4) 2.7[m]. P4.41 $x_f = 2.08[m]$. P4.42 $y(x) = \frac{t \sin(\theta_{max})}{v(\omega/v)x}$.

Solutions to problems

P4.1 When the $y$-axis points up, $a_y = -g$ and $v_{iy}$ is positive. The opposite applies when the $y$-axis is directed downward. The ball moves at the same horizontal speed as the cat; it’s always directly above the cat and splashes it when it comes back down. The cat is not happy about that.

P4.2 In each case have the sum of the forces is directed in the desired direction for $\vec{a}$.

P4.3 Calculate the momentum and energy using the formulas $||\vec{p}|| = m||\vec{v}||$ and $K = \frac{1}{2}mv^2$. Observe that two objects moving with equal momentum can carry different amounts of kinetic energy; this problem shows momentum and energy are different quantities.

P4.4 In each case, the sum $\vec{p}_A + \vec{p}_B$ after the separation equals the momentum of the station before the compartments split apart: $2m\vec{v} = m\vec{v}_A + m\vec{v}_B$.

P4.5 When there is a velocity, there is kinetic energy $K$. When the spring is stretched, there is spring potential energy $U_s$. When the position of the mass is above or below $y = 0$, there is gravitational potential energy $U_g$.

P4.6 The ball’s initial kinetic energy is the same on Earth and on the Moon. Because of conservation of energy, when the ball comes back to ground level, it will have the same kinetic energy as it had initially, regardless of the value of $g$.

P4.7 Define the zero potential-energy level to be at ground level. The bottom of the 10[m] pit has a lower potential energy on Earth because $g_{Earth} > g_{Moon}$. The ball will therefore gain more kinetic energy on Earth when it reaches the bottom of the pit and thus have a higher speed.

P4.8 The two balls have the same initial kinetic and initial potential energy. They also have the same final potential energy so they must have the same final kinetic energy. They will have the same speed when they hit the ground.

P4.9 The car’s weight contributes to the centripetal force at the top, while at the bottom the loop must exert a force to counteract the car’s weight to provide the same centripetal force and centripetal acceleration. Therefore, the normal force of the loop on the car is larger at the bottom.

P4.10 The rotation of the mass $M$ is at a constant angular velocity so the net torque on it is zero. Let us denote by $L_{rod}$, $L_{mass}$, and $L_{sys}$, the angular momenta of the rod, the mass $M$, and the total angular momentum of the mass-on-rod system. Initially, $L_{sys} = L_{rod} + L_{mass}$. When the mass $M$ detaches, its velocity $\vec{v}$ will remain the same as before the moment it detached. This means its angular momentum $L_{mass}$ will remain the same after it detaches. This in turn implies the rod will maintain its angular momentum too so its angular velocity will remain $\omega$.

P4.11 The period $T$ is not related to the maximum angle of oscillation $\theta_{max}$. The clocks will have the same $T$ if their strings have the same length.

P4.12 Using the formula $T = 2\pi\sqrt{\frac{L}{g}}$ and $g = 9.81[m/s^2]$ on Earth, we find $T_1 = 1.42[s]$, $T_2 = 1.74[s]$, and $T_3 = 2.01[s]$.

P4.13 The gravitational acceleration on Earth is stronger than on the Moon so the pendulum has a greater potential energy when deviated by an angle $\theta_{max}$ on Earth. Specifically, $U_{gy} = mg(1 - \cos(\theta_{max}))$. When it swings to $\theta = 0$ the pendulum will gain more kinetic energy on Earth, and hence it has a higher speed on Earth.

P4.14 The diver is rotating so we must consider the rotational kinetic energy $K_r$ and the angular momentum $L$ associated with this rotation. The kinetic energy $K$ and linear momentum $\vec{p}$ exist every time there is linear motion. Finally, we must also consider the potential energy $U_g$ since it is a part of the physics model for objects moving in a gravitational field.

P4.15 Because $T = 2\pi\sqrt{\frac{L}{g}}$, the period will decrease as the string shortens.

P4.16 First calculate the normal force $N$ between the floor on the box. Next calculate the force of friction $F_{friction} = \mu_k N$. For the box to move at constant velocity, you must counteract the kinetic force of friction and make the net force on the box zero. Therefore, $F_{push} = F_{friction} = 4.71[N]$.

P4.17 The energy that is lost is the kinetic energy. Before you catch the frisbee it has $K_i = \frac{1}{2}m v^2 = 19.7[J]$ of kinetic energy. After you stop it with your hand, the frisbee has $K_f = 0$. The change in $K$ before and after the frisbee is caught is $K_f - K_i = -19.7[J] = E_{lost} = 19.7[J]$.

P4.18 (1) The momentum of the players after the collision is zero, therefore the sum of their momenta before the collision must also be zero: $m_1 v_1 + m_2 v_2 = 0$. We can find $v_2$ since $m_1$, $m_2$, and $v_1$ are known. (2) Since the players come to a stop after the collision, they lose all their kinetic energy, thus the collision is not elastic. In fact, the collision is completely inelastic.

P4.19 (1) Build a force diagram and include the weights, tension forces, and the force exerted by the professor. The total mass of the system is $2m = 40[kg]$.
(2) Use the 4th equation of motion to calculate the acceleration. (3) Apply Newton’s 2nd law \( F = ma \) for the system.

**P4.20** The work done to compress the spring transforms into kinetic energy. Equate the kinetic energy with the work stored in the spring then compare it to the kinetic energy given to the mass in the first case: \( W_a[1] \) turns into speed \( v[m/s] \), while \( 2W_a[1] \) turns into speed \( \sqrt{2}v [m/s] \), because \( K = \frac{1}{2}mv^2[3] \).

**P4.21** Use the initial and final angular velocities and the equation \( \omega(t) = \omega_i + at \) to calculate acceleration then find the time using \( T = I\alpha \). The work done by you is equal to the rotational kinetic energy gained by the door \( K = \frac{1}{2}I(\omega_f)^2 \).

**P4.22** The spring’s potential energy is equal to the sum of the linear and rotational kinetic energies of the ball: \( \frac{1}{2}k(\Delta x)^2 = \frac{1}{2}mv^2 + \frac{1}{2}I_\text{ball}\omega^2 \). Because the ball is rolling without skidding, we can calculate the linear velocity from the angular velocity and the radius \( v = \omega R = 3[m/s] \). Substituting all knowns into the energy equation, we can solve for the mass \( m \).

**P4.23** Calculate the acceleration of the two blocks if they were to move together, then calculate the friction force required to provide that acceleration to the upper block, finally solve for \( \mu_k = \frac{F_f}{N} \) where \( N = m_1g \).

**P4.24** Find the time \( t = t_{hit} \) when the ball reaches height \( \frac{1}{2}t \), then plug that time into the position equations of the shell so that \( x_{shell}(t_{hit}) = 0 + v_i \cos(45°)t_{hit} = d \) and \( y_{shell}(t_{hit}) = 0 + v_i \sin(45°)t_{hit} - \frac{1}{2}gt^2 = \frac{d}{2} \).

**P4.25** Draw a force diagram for each block considering that acceleration is zero, then construct three equations for the three blocks and calculate \( \mu_k \). For part (3), you can re-use the force diagram but this time the acceleration is not zero. The three blocks have the same acceleration.

**P4.26** To answer (1), find the maximum range of the first ball. Half of this range is the distance \( d \) where you should fire the second ball. To answer (2), find the maximum height reached by the first ball then use conservation of energy to find the initial velocity needed for the second ball.

**P4.27** We must calculate the lift force \( F_{lift} \) whose horizontal (radial) component provides the required centripetal acceleration: \( F_{lift} \sin 30° = F_r = m \frac{v^2}{R} = 1400\left(\cos 5°/3.6\right)^2 \), so \( F_{lift} = 158 \times 10^3[N] \). The vertical component of the lift force is \( F_{lift} \cos 30° = 137.2 \times 10^3[N] \) which equals the weight of the plane \( mg \), thus the plane has no acceleration in the vertical direction.

**P4.28** The bottle has linear kinetic energy \( K_i = \frac{1}{2}m(12.5)^2[3] \) before the subway starts braking. This energy transforms into both linear kinetic and rotational kinetic energy of the rolling bottle. Equating the energy before and after stopping, we obtain \( \frac{3}{2}m(12.5)^2 = \frac{1}{2}mv^2 + \frac{1}{2}(I_{bottle})\omega^2 \). Cancelling the mass \( m \) and replacing the angular velocity \( \omega \) using \( \omega = \frac{v}{R} \), we can solve for \( v \). We find \( v = 10.2[m/s] \).

**P4.29** (1) Using conservation of momentum \( p_i = p_f \) construct two equations relating \( v_i, v_1, \) and \( v_2 \). Calculate the minimum \( v_2 \) that gets puck 2 to the pocket (such that puck 2 stops when it reaches the pocket) using the relation between work done by the force of friction and the kinetic energy of the puck \( \Delta K = W \), then find \( v_1 \) and \( v_1 \) using the previous equations. (2) Plug in the provided values.

(3) Use \( K_f = K_f + W_{\text{out}} \) with \( K_f = 0 \). (4) Calculate \( K_{1,i} \) then compare it to \( K_{i+1} + K_{2,f} \) after the collision.

**P4.30** Use \( a = r\alpha \) to find \( \alpha \), then use \( T = I\alpha \) to find the torque. Use the angular equations of motion to find \( \theta(4) \). The number of revolutions is \( \frac{\theta(4)}{2\pi} \).

**P4.32** First we use \( v_i = K_f \) for the pendulum, obtaining \( MgL = \frac{1}{2}Mv_{in}^2 \) and thus \( v_{out} = \sqrt{2gL} \). Next we use a momentum reasoning \( p_{in} = p_{out} \) where the incoming momentum is that of the mass \( M \) and the outgoing momentum is that of the mass \( m \). The conservation of momentum equation becomes \( Mv_{in} + 0 = 0 + mv_{out} \), where \( v_{out} \) is the velocity of the mass \( m \) after the collision, and the momentum of the pendulum is zero after the collision since it doesn’t bounce back. Solving for \( v_{out} \) we find \( v_{out} = \frac{M}{m} \sqrt{2gL} \). Finally, we use an energy calculation \( K_i = W_{\text{out}} \), which becomes \( \frac{1}{2}m \left( \frac{M}{m} \sqrt{2gL} \right)^2 = mg\mu_k d \). After some simplifications, we find \( \mu_k = \frac{M^2}{m^2} \frac{g}{2L} \).

**P4.33** We want to find the range—how far the ball will reach after being kicked—in both situations. The first thing to calculate is the total time of flight by solving for \( t \): in \( 0 = 0 + v_{y,i}t + \frac{1}{2}(-g)t^2 \). The time of flight will be \( 4.347[s] \) on the Nevado Huascarán summit, and \( 4.316[s] \) on the North Pole. The range in each case corresponds to \( d = v_{x,i}4.347 = 92.21[m] \) and \( d = v_{y,i}4.316 = 91.56[m] \). The difference in range is \( 92.21 - 91.56 = 0.65[m] \).

**P4.34** This is a kinematics question. Start from the equation \( v(t) = at + v_i \) and \( a = -9.81 \). We know \( v(t_{top}) = 0 \), so we can solve to find \( t_{top} \).

**P4.36** The normal force between the slug and the turntable is \( N = mg \). With the slug located at radius \( R \), the centripetal acceleration required to keep the slug on the disk is \( F_c = ma_c = m(\frac{v}{R})^2 \). The friction force available is \( F_T = 0.4mg \). The slug will fly off when the friction force becomes insufficient, which happens at a distance of \( R = \frac{4a_y g}{v^2} \) from the centre.

**P4.37** The equation for \( F_{fs} \) is \( F_{fs} = \mu_k N \), where \( N = mg \) is the normal force (the contact force between the fridge and the elevator floor). The force diagram on the elevator reads \( \sum F_y = N - mg = ma_y \). When the elevator is static, \( a_y = 0 \) so \( N = mg \). If \( a_y > 0 \) (upward acceleration), then we must have \( N > mg \); hence the friction force will be larger than when the elevator is static. When \( a_y < 0 \) (downward acceleration), \( N \) must be smaller than \( mg \), and consequently there will be less \( F_{fs} \).

**P4.38** The coin furthest from the centre will be the first to fly off the spinning turntable because the centripetal force required to keep this coin turning is the largest. Recall that \( F_c = ma_c \), that \( a_c = \frac{v^2}{R} \), and that \( v = \omega R \). If the turntable turns with angular velocity \( \omega \), the centripetal acceleration required to keep a coin turning in a radius \( R \) is \( F_c = mw^2R \). This centripetal force must be supplied by the static force of friction \( F_{fs} \) between the coin and the turntable. Larger \( R \)s require more \( F_{fs} \).
The two pulleys have the same moment of inertia $I$. The string wound around the larger radius $R$ will produce the larger torque. Higher torque will produce more angular acceleration and therefore a bigger angular velocity and angular kinetic energy.

The friction force is proportional to the normal force. The friction on each side of each disk is $F_f = 0.3 \times 5000 = 1500[N]$ for a total friction force of $F_f = 3000[N]$ per wheel. (2) The friction force of the brakes acts with a leverage of 0.06[m], so the torque produced by each brake is $T = 0.06 \times 3000 = 180[N\cdot m]$. (3) The kinetic energy of a 100[kg] object moving at 10[m/s] is equal to $K = \frac{1}{2}100(10)^2 = 5000[J]$. We’ll use $K_f = W = \theta$, where $W$ is the work done by the brakes. Let $\theta_{stop}$ be the angle of rotation of the tires when the brake stops. The work done by each brake is $180\theta_{stop}$. It will take a total of $\theta_{stop} = \frac{3000}{180} = 138[\text{rad}]$ to stop the angle. This corresponds to 2.21 turns of the wheels! (4) Your stopping distance will be $13.5 \times 0.20 = 2.7[m]$. Yay for disk brakes!

The energy equation $\sum E_i = \sum E_f$ in this case is $U_i = U_f + K_f$, or $mg(6-6\cos 50^\circ) = mg(6-6\cos 10^\circ) + \frac{1}{2}mv^2$, which can be simplified to $v^2 = 12g(\cos 10^\circ - \cos 50^\circ)$. Solving for $v$ we find $v = 4.48[m/s]$. Now for the projectile motion part. The initial velocity is 4.48[m/s] at an angle of 10° with respect to the horizontal, so $v_i = (4.42, 0.778)[m/s]$. Tarzan’s initial position is $(x_i, y_i) = (6\sin(10), 6[1 - \cos(10)]) = (1.04, 0.0991)[m]$. To find the total time of flight, we solve for $t$ in $0 = -4.9t^2 + 0.778t + 0.0991$ and find $t = 0.2373[m]$. Tarzan will land at $x_f = 6\sin(10) + 4.42 = 2.08[m]$.

We begin writing by the general equation of motion for a pendulum: $	heta(t) = \theta_{\text{max}} \cos(\omega t)$, where $\omega = \sqrt{g/L}$. Enter the walkway, which is moving to the left at velocity $v$. If we choose the $x = 0$ coordinate at a time when $\theta(t) = \theta_{\text{max}}$, the pattern on the walkway can be described by the equation $y(x) = \ell \sin(\theta_{\text{max}}) \cos(kx)$, where $k = 2\pi/L$, and $\ell$ tells us how long (measured as a distance in the $x$-direction) it takes the pendulum for the pendulum to complete one cycle. One full swing of the bucket takes $T = 2\pi/\omega[t]$. In that time, the moving walkway will have moved a distance of $vT$ metres. So one cycle in space (one wavelength) is $\lambda = vT = 2\pi/\omega$. We conclude that the equation of the point on the moving sidewalk is $y(x) = \ell \sin(\theta_{\text{max}}) \cos(\omega v x)$.

P5.114 $f(x) = \sum_{n=0}^{\infty} x^n$. P5.115 (a) $\sum_{n=0}^{\infty} (m+1)x^n$. (b) $\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$. (c) $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$. (d) $\sum_{n=0}^{\infty} \frac{x^n}{n^2}$. (e) $\sum_{n=0}^{\infty} \frac{x^n}{n(n+1)!}$. (f) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. (g) $g(x) = x \tan(x)$. (h) $g(x) = \sec(x)$. (i) $g(x) = e^x$. (j) $g(x) = \sin(x)$. (k) $g(x) = \cos(x)$. (l) $g(x) = \tan(x)$. (m) $g(x) = \sec(x)$. (n) $g(x) = \csc(x)$. (o) $g(x) = \cot(x)$.

Solutions to selected problems

P5.1 (10) For $x = 5$, we have $\lim_{x \to 5} f(x) \neq f(5)$ so the function $f(x)$ is discontinuous at $x = 5$. This is called a removable discontinuity. (11) The function is continuous everywhere except for the discontinuities. The function $f(x)$ is continuous from the right at $x = 5$ and $x = 2$ so these endpoints are included in the intervals to the right.

P5.3 We can choose the value $\delta$ (how close we must be to $x = 5$) as a function of the precision $\epsilon$ specified by the skeptic. One possible choice is $\delta = \frac{\epsilon}{5}$ although $\frac{1}{3}$ or $\frac{1}{5}$ would also work. First we prove $\lim_{x \to 3} x = 15$ using the following chain of inequalities. Starting from the assumption $x \in [5, 5 + \delta)$, we have

\[
5 \leq x < 5 + \delta
\]

Thus we have shown that $[32 - 15] < \epsilon$ for all possible $\epsilon$ which is a proof of $\lim_{x \to 3} x = 15$. The procedure is similar for the limit from the left: starting from the assumption $x \in (5 - \delta, 5]$ and choosing $\delta = \frac{1}{5}$ again, we show that $[33 - 15] < \epsilon$, which is a proof of $\lim_{x \to 3} x = 15$.


P5.50 The derivative is $\frac{dy}{dx} = 100 \cos(\theta - 15^\circ)$. The maximum slope is when $\theta = 15^\circ$ and is equal to $y'(15^\circ) = 100$. When $\theta = 75^\circ$ the slope is $100 \cos(75^\circ - 15^\circ) = 100 \cos(60^\circ) = 50$, which is half of the maximum.

P5.51 First take the derivative with respect to $x$ of the entire equation: $\frac{dy}{dx} = \frac{dy}{dz}[4]$. We obtain the equation $2x + 2y \frac{dy}{dx} = 0$, from which we conclude $\frac{dy}{dx} = -\frac{x}{y}$. The condition $\frac{dy}{dx} = 1$ is equivalent to the line $y = x$, which intersects the circle $x^2 + y^2 = 4$ at points $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$.

P5.52 Taking the implicit derivative of the equation (which is an ellipse), we find $\frac{dy}{dx} = \frac{-x}{y}$. Isolating the derivative, we find $\frac{dy}{dx} = \frac{-x}{y}$. Note the similarity with the slope equation of a circle. When $x = 0$, $\frac{dy}{dx} = 0$. The vertical line $x = 1$ intersects the curve in two places: the slope is positive $\frac{dy}{dx} = \frac{-1}{2\sqrt{3}}$ on the bottom half of the ellipse, and the slope is negative $\frac{dy}{dx} = \frac{-1}{2\sqrt{3}}$ on the top half.
P5.75 The area of a rectangle is width times height: \( A = w\ell \). Assuming we use all the fence, the rectangle constructed will have perimeter equal to \( p = 2w + 2\ell \). We can rewrite the width as a function of the (fixed) perimeter and the length: \( w = \frac{p}{2} - \ell \). To find the maximum area, we have to maximize the function \( A(\ell) = w\ell = \left(\frac{p}{2} - \ell\right) \ell \). \( A'(\ell) = -\ell + \left(\frac{p}{2} - \ell\right) \). Solving for \( \ell \) in \( A'(\ell) = 0 \) we find \( \ell = \frac{p}{4} \).
The largest-area rectangle with perimeter \( p \) is a square with side length \( \frac{p}{4} \).

P5.82 This is the fundamental theorem of calculus, which states that the derivative of the integral function of a function is the function itself.

P5.96 (a) We’re looking for a function \( y(x) \) whose derivative is equal to \( \frac{1}{2} x \). Integrating we find \( y(x) = \frac{1}{2} x^2 \), but there could also have been an additive constant \( C \) so the general solution is \( y(x) = \frac{1}{2} x^2 + C \). (b) Taking the indefinite integral on both sides of the equation \( \frac{dy}{dx} = \cos x \) we find \( y = \sin x + C \). (c) Integrating \( \frac{dy}{dx} = 2x + 3 \) we find \( y = x^2 + 3x + C \).

P5.97 For a second-order differential equation (a differential equation involving second derivatives) there will be two two independent solutions. The hint tells us these solutions are \( e^{-\lambda x} \) and \( xe^{-\lambda x} \) and we choose \( \lambda = 1 \) to satisfy the differential equation. The general solution is any linear combination of these solutions \( f(x) = C_1 e^{-x} + C_2 xe^{-x} \), where \( C_1 \) and \( C_2 \) are arbitrary constants. By computing \( f(0) \) and \( f'(0) \), we find that the choice \( C_1 = 1 \) and \( C_2 = 2 \) satisfies the initial conditions \( f(0) = 1 \) and \( f'(0) = 1 \). See bit.ly/1kkzhno.

P5.98 The arc length of a curve \( f(x) \) is \( \ell = \int_0^b \sqrt{1 + (f'(x))^2} \, dx \). In the current problem \( f'(x) = x \) and the integral we want to find is \( I = \int \sqrt{1 + x^2} \, dx \).
Note \( \sqrt{1 + x^2} = \frac{1 + x^2}{\sqrt{1 + x^2}} = \frac{1}{\sqrt{1 + x^2}} + x \). The first term is the derivative of \( \sinh^{-1}(x) \). For the second term, use integration by parts with \( u = x \), \( dv = \frac{1}{\sqrt{1 + x^2}} \, dx \). You’ll obtain the equation \( I = \sinh^{-1}(x) + x\sqrt{1 + x^2} - C \), where \( C \) is the integral we want to find.
Evaluating \( I \) at the endpoints we find \( \ell = I_1^2 = \frac{1}{2} \sinh^{-1}(1) + \frac{1}{2} \sqrt{2} \). If you obtained the answer \( \frac{1}{2} \ln(1 + \sqrt{2}) + \frac{1}{2} \sqrt{2} \), it’s also correct because \( \sinh^{-1}(x) \equiv \ln(x + \sqrt{1 + x^2}) \).

P5.99 We want to calculate the square of the voltage during one period \( T \):
\[
V_{avg}^2 = \int_0^T 155.57^2 \cos^2(\omega t) \, dt.
\]
Making the substitution \( r = \omega t \), we obtain an equivalent expression \( V_{avg}^2 = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} 155.57^2 \cos^2(\tau) \, d\tau \), which we can calculate
\[
155.57^2 \int_0^{\frac{\pi}{2}} \cos^2(\tau) \, d\tau = 155.57^2 \left[ \frac{\tau}{2} + \sin(\tau)\cos(\tau) \right]_0^\frac{\pi}{2} = 155.57^2 \left[ \frac{\pi}{4} \right] = 155.57^2.
\]
The root-mean-squared voltage is therefore \( V_{rms} = \sqrt{\frac{155.57^2}{2}} = 155.57 \sqrt{2} = 110[V] \).

P5.104 Observe that \( \frac{1}{n^{2/n}} < \frac{1}{n} \) for all \( n \geq 1 \). We know that \( \sum_{n=1}^{\infty} \frac{1}{n} \) converges since it is a geometric series with \( r = \frac{1}{2} \) and \( |r| < 1 \). By the direct comparison test, \( \sum_{n=1}^{\infty} \frac{1}{n^{2/n}} \) also converges.

P5.105 The limit of the \( n^{th} \) term in the series is \( \lim_{n \to \infty} \frac{n^2}{2n^2 + 1} = \frac{1}{2} \neq 0 \). The series \( \sum_{n=1}^{\infty} \frac{n^2}{2n^2 + 1} \) involves a limit of an infinite number of nonzero terms therefore it must be divergent.

P5.109 This is a geometric series with \( r = \frac{1}{2} \) and \( a = \frac{2}{3} \) so the infinite sum is
\[
\frac{a}{1-r} = \frac{\frac{2}{3}}{1-\frac{1}{2}} = \frac{4}{1} = 4.
\]

P5.110 In each case we use the formula for the geometric series \( \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \).

P5.111 Use the geometric series formula \( \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \). bit.ly/1e9FS2v